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A New Modification of the Weibull Distribution: Model, Theory, and Analyzing Engineering Data Sets

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Abstract: Symmetrical as well as asymmetrical statistical models play a prominent role in describing and predicting the real-world phenomena of nature. Among other fields, these models are very useful for modeling data in the sector of civil engineering. Due to the applicability of the statistical models in civil engineering and other related sectors, this paper offers a statistical methodology to improve the distributional flexibility of traditional models. The suggested method/approach is called the extended-X family of distributions. The proposed method has the ability to generate symmetrical and asymmetrical probability distributions. Based on the extended-X family approach, an updated version of the Weibull model, namely, the extended Weibull model, is studied. The proposed model is very flexible and has the ability to capture the symmetrical and asymmetrical shapes of its density function. For the extended-X method, the estimation of parameters, a simulation study, and some mathematical properties are derived. Finally, the practical illustration/usefulness of the suggested model is shown by analyzing two data sets taken from the field of engineering. Both data sets represent the fracture toughness of alumina (Al_2O_3).

Keywords: Weibull model; family of distributions; goodness of fit measures; maximum likelihood estimation; simulation study; statistical modeling; alumina

Citation: Alshanbari, H.M.; Ahmad, Z.; El-Bagoury, A.A.-A.H.; Odhah, O.H.; Rao, G.S. A New Modification of the Weibull Distribution: Model, Theory, and Analyzing Engineering Data Sets. *Symmetry* **2024**, *16*, 611. https://doi.org/10.3390/sym16050611

Academic Editor: Markus Meringer

Received: 3 August 2023
Revised: 25 April 2024
Accepted: 29 April 2024
Published: 15 May 2024

1. Introduction

Probability distributions are widely implemented in almost every field, especially in civil engineering, healthcare sciences, electrical engineering, corrosion, aerospace, management, hydrology, and financial sectors, among others. For more information about the implementation of the probability distributions, we refer to [1–7].

Undoubtedly, probability distributions play a significant and important role in modeling real-life scenarios in every field of life. However, it is also a crystal clear fact that no specific probability distribution can provide reasonably a good fit in all scenarios. Therefore, we often need to have probability distributions with updated distributional flexibility to fit the practical data sets closely. The need to optimally fit real data sets in different scenarios has led researchers to explore new probability distributions. To date, a substantial number of papers have appeared in the literature focusing on the development of new probability distributions (or new distributional methods or family of distributions) to exceptionally fit the practical data sets [8–14].

The modified/updated probability distributions as well as the new distributional methods are introduced by incorporating different parameters such as the transmuter parameter, scale parameter, location parameter, or rate parameter. Thanks to these probability distributions and the family of probability distributions, the majority of them have carried out the inevitable goal of optimally fitting practical data sets. On the other hand, however, the number of parameters of these distributional methods has also increased to seven [15].

Indubitably, the addition of new parameters significantly improves the fitting ability of the existing distributions; however, it also leads to some problems such as the (i) estimation consequences, (ii) the cumbersome process of deriving the distribution characteristics, and (iii) re-parameterization problems, etc. This paper also contributes to the literature by considering and implementing a useful method, namely, the extended-X (E-X) method for updating the distributional flexibility and characteristics of the probability distributions. Unlike some other traditional distributional methods, the E-X method has a single additional parameter rather than two, three, or more additional parameters. Below, we provide the basic distributional functions of the E-X method.

Definition 1. *The distribution function (DF) $F(x;\alpha,\boldsymbol{\xi})$ of the E-X method is defined by*

$$F(x;\alpha,\boldsymbol{\xi}) = \frac{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}}{e^{\alpha}-1}, \qquad x \in \mathbb{R}, \tag{1}$$

where $G(x;\boldsymbol{\xi})$ is a valid DF with a vector of parameters $\boldsymbol{\xi}$ and $\alpha \in \mathbb{R}^+$ is an additional parameter.

Corresponding to $F(x;\alpha,\boldsymbol{\xi})$, the probability density function (PDF) $f(x;\alpha,\boldsymbol{\xi})$, survival function (SF) $S(x;\alpha,\boldsymbol{\xi})$, hazard function (HF) $h(x;\alpha,\boldsymbol{\xi})$, reverse HF $r(x;\alpha,\boldsymbol{\xi})$, and cumulative HF $H(x;\alpha,\boldsymbol{\xi})$ are given by

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})}{e^{\alpha}-1} e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}, \qquad x \in \mathbb{R}, \tag{2}$$

$$S(x;\alpha,\boldsymbol{\xi}) = \frac{e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)} - 1}{e^{\alpha}-1}, \qquad x \in \mathbb{R},$$

$$h(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})}{e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)} - 1} e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}, \qquad x \in \mathbb{R},$$

$$r(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})}{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}} e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}, \qquad x \in \mathbb{R},$$

and

$$H(x;\alpha,\boldsymbol{\xi}) = -\log\left(\frac{e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)} - 1}{e^{\alpha}-1}\right), \qquad x \in \mathbb{R},$$

respectively.

The new PDF presented in Equation (2) is most tractable when the baseline PDF $g(x;\alpha,\boldsymbol{\xi})$ and DF $G(x;\alpha,\boldsymbol{\xi})$ have simple analytical expressions.

Some key motivations/advantages of implementing the E-X method are the following:

- The E-X is a prominent method to obtain flexible models that are capable of capturing different patterns of $f(x;\alpha,\boldsymbol{\xi})$ and $h(x;\alpha,\boldsymbol{\xi})$.
- The E-X approach is capable of updating the distribution flexibility of the baseline models to provide a close fit to real-world data sets.
- The E-X method generates new models having a closed form of $F(x;\alpha,\boldsymbol{\xi})$.
- The quantile function (QF) of the E-X method is in an explicit form, which makes it easy to generate random numbers without using the rootSolve function in the R programming software.
- The E-X approach adds only one additional parameter to introduce newly updated distributions, rather than adding two or more additional parameters.

In Section 2, we discuss a special member of the E-X method. For the new model, the expressions of $F(x;\alpha,\boldsymbol{\xi})$, $f(x;\alpha,\boldsymbol{\xi})$, $S(x;\alpha,\boldsymbol{\xi})$, and $h(x;\alpha,\boldsymbol{\xi})$ are obtained. In addition to the numerical expressions, a visual display of $f(x;\alpha,\boldsymbol{\xi})$ and $h(x;\alpha,\boldsymbol{\xi})$ is also provided. Section 3 provides certain distributional properties of the E-X distributions. The estimation and simulation studies are carried out in Section 4. Two practical data sets are analyzed in Section 5. The future research directions are discussed in Section 6. Some final remarks are provided in Section 7.

2. A Sub-Model Description and its Special Cases

This section is devoted to introducing a special sub-model of the E-X family, called the extended Weibull (E-Weibull) distribution. Furthermore, the special cases of the E-Weibull are also discussed.

2.1. A Sub-Model Description

For $x \in \mathbb{R}^+$, let $G(x;\boldsymbol{\xi})$, $g(x;\boldsymbol{\xi})$, and $h(x;\boldsymbol{\xi})$ be the DF, PDF, and HF of the two parameters ($\gamma \in \mathbb{R}^+,\ \theta \in \mathbb{R}^+$) Weibull distribution [16], given by

$$G(x;\boldsymbol{\xi}) = 1 - e^{-\gamma x^{\theta}}, \qquad x \in \mathbb{R}^+, \tag{3}$$

$$g(x;\boldsymbol{\xi}) = \theta\gamma x^{\theta-1} e^{-\gamma x^{\theta}}, \qquad x \in \mathbb{R}^+,$$

and

$$h(x;\boldsymbol{\xi}) = \theta\gamma x^{\theta-1}, \qquad x \in \mathbb{R}^+,$$

respectively, where $\boldsymbol{\xi} = (\gamma,\theta)$. The Weibull distribution reduces to the (i) Rayleigh distribution when $\theta = 2$ in Equation (3), and (ii) exponential distribution when $\theta = 1$ in Equation (3).

Some possible plots of $h(x;\boldsymbol{\xi})$ of the Weibull distribution are presented in Figure 1. These plots show that the HF of the Weibull distribution can either be (i) increasing, when $\theta > 1$ (with any value of $\gamma \in \mathbb{R}^+$), (ii) decreasing, when $\theta < 1$ (with any value of $\gamma \in \mathbb{R}^+$), or (iii) constant, when $\theta = 1$ (with any value of $\gamma \in \mathbb{R}^+$).

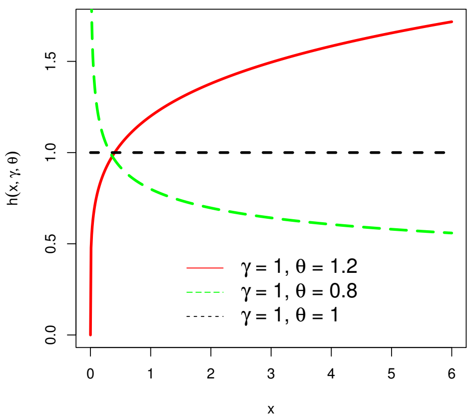

Figure 1. Visual display of $h(x;\boldsymbol{\xi})$ for $\gamma = 1$ and different values of θ.

Figure 1 shows that the HF in the Weibull distribution is only able to capture monotonic shapes such as increasing, decreasing, or constant. Therefore, in most cases where the HF of the data has non-monotonic behavior (such as unimodal, modified unimodal, or bathtub shapes), particularly when data follows a bathtub behavior, the Weibull distribution does not provide the best fit [17–21]. To overcome this deficiency of $h(x;\boldsymbol{\xi})$ of the Weibull distribution, we introduce a modified/updated version of the Weibull distribution with the following DF $F(x;\alpha,\boldsymbol{\xi})$.

$$F(x;\alpha,\boldsymbol{\xi}) = \frac{e^{\alpha} - e^{\alpha\left(1-\left[1-e^{-\gamma x^{\theta}}\right]^2\right)}}{e^{\alpha} - 1}, \qquad x \in \mathbb{R}^+, \alpha \in \mathbb{R}^+, \gamma \in \mathbb{R}^+, \theta \in \mathbb{R}^+, \tag{4}$$

and SF $S(x;\alpha,\boldsymbol{\xi})$ are given by

$$S(x;\alpha,\boldsymbol{\xi}) = \frac{e^{\alpha\left(1-\left[1-e^{-\gamma x^{\theta}}\right]^{2}\right)}-1}{e^{\alpha}-1}.$$

The visual illustrations of $F(x;\alpha,\boldsymbol{\xi})$ and $S(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution are presented in Figure 2.

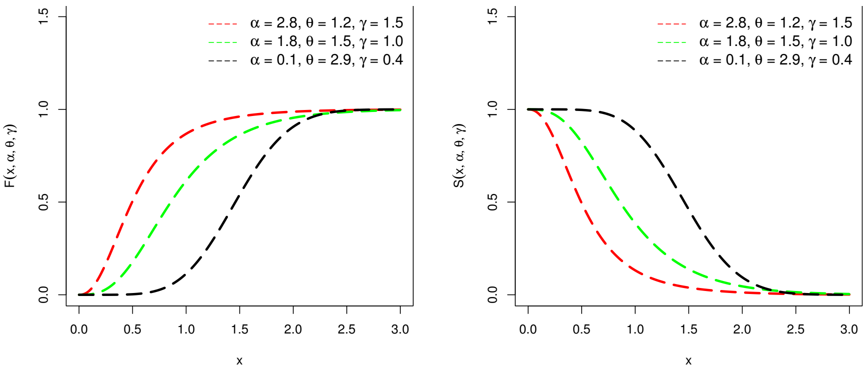

Figure 2. A visual display of $F(x;\alpha,\boldsymbol{\xi})$ and $S(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution.

For $x \in \mathbb{R}^{+}, \alpha \in \mathbb{R}^{+}, \gamma \in \mathbb{R}^{+}$, and $\theta \in \mathbb{R}^{+}$, the PDF $f(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution is given by

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2\gamma\theta\alpha x^{\theta-1}e^{-\gamma x^{\theta}}\left[1-e^{-\gamma x^{\theta}}\right]}{(e^{\alpha}-1)}e^{\alpha\left(1-\left[1-e^{-\gamma x^{\theta}}\right]^{2}\right)}. \quad (5)$$

The corresponding HF $h(x;\alpha,\boldsymbol{\xi})$ is

$$h(x;\alpha,\boldsymbol{\xi}) = \frac{2\gamma\theta\alpha x^{\theta-1}e^{-\gamma x^{\theta}}\left[1-e^{-\gamma x^{\theta}}\right]}{e^{\alpha\left(1-\left[1-e^{-\gamma x^{\theta}}\right]^{2}\right)}-1}e^{\alpha\left(1-\left[1-e^{-\gamma x^{\theta}}\right]^{2}\right)}.$$

A visual illustration of $f(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution is shown in Figure 3. The visual illustration of $f(x;\alpha,\boldsymbol{\xi})$ is provided for (i) $\alpha = 1.8, \theta = 0.5, \gamma = 1.0$, (red-line curve), (ii) $\alpha = 2.5, \theta = 2.2, \gamma = 1.0$, (green-line curve), (iii) $\alpha = 0.01, \theta = 3.6, \gamma = 0.1$, (black-line curve), and (iv) $\alpha = 0.01, \theta = 2.9, \gamma = 0.4$, (blue-line curve). Figure 3 shows that the PDF of the E-Weibull distribution has four different shapes, such as (i) decreasing (red-line curve), (ii) right-skewed (green-line curve), (iii) left-skewed (black-line curve), and (iv) symmetrical (blue-line curve).

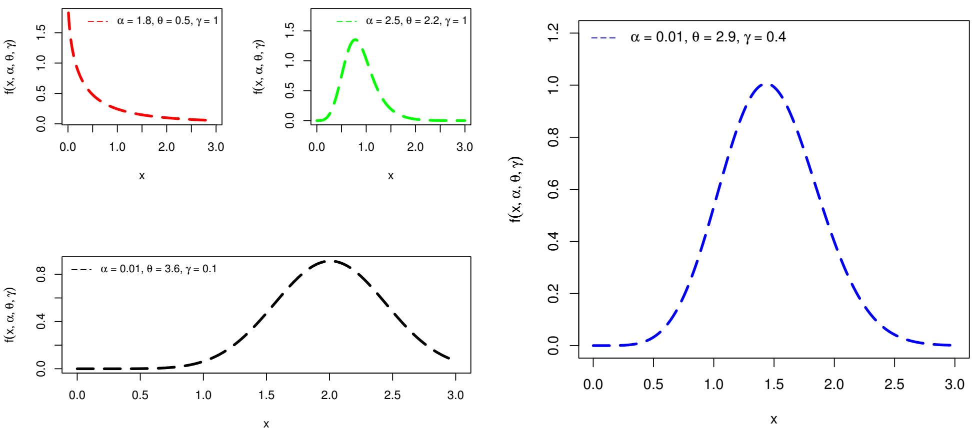

Figure 3. A visual display of $f(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution.

A visual illustration of $h(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution is provided in Figure 4. The visual illustrations of $h(x;\alpha,\boldsymbol{\xi})$ are sketched for (i) $\alpha = 2.1, \theta = 0.8, \gamma = 1.1$, (red-line curve), (ii) $\alpha = 1.2, \theta = 0.5, \gamma = 1.0$, (green-line curve), (iii) $\alpha = 12.2, \theta = 2.5, \gamma = 1.4$, (black-line curve), (iv) $\alpha = 0.8, \theta = 1.7, \gamma = 2$, (blue-line curve), and (v) $\alpha = 2.1, \theta = 0.3$, $\gamma = 0.2$, (gold-line curve). Figure 4 shows that the HF of the E-Weibull distribution has five different shapes, such as (i) uni-modal (red-line curve), (ii) decreasing (green-line curve), (iii) increasing–decreasing–increasing or modified uni-modal (black-line curve), (iv) increasing (blue-line curve), and (v) bathtub (gold-line curve).

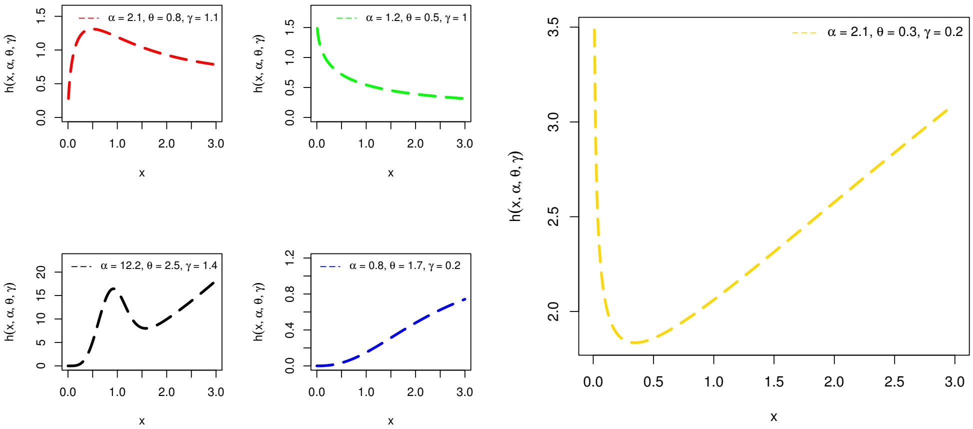

Figure 4. A visual display of $h(x;\alpha,\boldsymbol{\xi})$ of the E-Weibull distribution.

2.2. Special Cases of the E-Weibull Distribution

This subsection offers the special cases of the E-Weibull distribution. The E-Weibull distribution can be reduced to five new subcases. Let X have the E-Weibull distribution with DF in Equation (4), then X is reduced.

3. The Statistical Properties

Here, we derive some statistical properties (SPs) of the E-X distributions such as series representation, quantile function (QF), rth moment, and moment generating function (MGF).

3.1. The Series Representation

This subsection offers a series representation of $f(x;\alpha,\boldsymbol{\xi})$ of the E-X distributions. Consider the series e^x, we have

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} .$$

By incorporating the above series in Equation (2), we obtain

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2}{(e^\alpha - 1)} \sum_{k=0}^{\infty} \frac{\alpha^{k+1}}{k!} g(x;\boldsymbol{\xi}) G(x;\boldsymbol{\xi}) \left(1 - [G(x;\boldsymbol{\xi})]^2\right)^k. \quad (6)$$

Using the series

$$(1-z)^k = \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} z^i, \qquad |z| < 1.$$

Thus, from Equation (6), we obtain

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2}{(e^\alpha - 1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!} g(x;\boldsymbol{\xi}) [G(x;\boldsymbol{\xi})]^{2i+1}. \quad (7)$$

The form of $f(x;\alpha,\boldsymbol{\xi})$ provided in Equation (7) can also be expressed as

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!} \Delta_i(x;\boldsymbol{\xi}), \tag{8}$$

where $\Delta_i(x;\boldsymbol{\xi}) = g(x;\boldsymbol{\xi})[G(x;\boldsymbol{\xi})]^{2i+1}$.

3.2. The QF

The QF plays a useful role in generating random numbers from a probability distribution. The QF of E-X distributions, denoted by Q_u, has the following form

$$x_q = G^{-1}\left(1 - \frac{\log(e^{\alpha} - u[e^{\alpha}-1])}{\alpha}\right)^{\frac{1}{2}}, \tag{9}$$

where $u \in (0,1)$.

3.3. The rth Moment

The rth moment of the E-X distributions with PDF $f(x;\alpha,\boldsymbol{\xi})$, denoted by μ'_r, is derived as

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\alpha,\boldsymbol{\xi}) dx. \tag{10}$$

Using Equation (8) in Equation (10), we obtain

$$\mu'_r = \frac{2}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!} \int_{-\infty}^{\infty} x^r g(x;\boldsymbol{\xi})[G(x;\boldsymbol{\xi})]^{2i+1} dx,$$

$$\mu'_r = \frac{2}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!} \frac{2i+2}{2i+2} \int_{-\infty}^{\infty} x^r g(x;\boldsymbol{\xi})[G(x;\boldsymbol{\xi})]^{(2i+2)-1} dx,$$

$$\mu'_r = \frac{2}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!(2i+2)} \int_{-\infty}^{\infty} x^r (2i+2) g(x;\boldsymbol{\xi})[G(x;\boldsymbol{\xi})]^{(2i+2)-1} dx,$$

$$\mu'_r = \frac{(i+1)^{-1}}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!(2i+2)} \int_{-\infty}^{\infty} x^r m_i(x;\boldsymbol{\xi}) dx, \tag{11}$$

where $m_i(x;\boldsymbol{\xi}) = (2i+2)g(x;\boldsymbol{\xi})[G(x;\boldsymbol{\xi})]^{(2i+2)-1}$ is the exponentiated PDF with exponentiated parameter $(2i+2)$. We can also express Equation (11) as follows

$$\mu'_r = \frac{(i+1)^{-1}}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1}}{k!(2i+2)} A_{r,i}(x;\boldsymbol{\xi}), \tag{12}$$

where

$$A_{r,i}(x;\boldsymbol{\xi}) = \int_{-\infty}^{\infty} x^r m_i(x;\boldsymbol{\xi}) dx.$$

For $r = 1$ and $r = 2$, we obtain the mean and variance for any sub-model of the E-X family. Using Equation (12), we can derive the rth moment for any sub-model of the proposed class. Furthermore, the MGF of X, expressed by $M_X(t)$, is given by

$$M_X(t) = \frac{(i+1)^{-1}}{(e^{\alpha}-1)} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} (-1)^i \binom{k}{i} \frac{\alpha^{k+1} t^r}{k! r! (2i+2)} A_{r,i}(x;\boldsymbol{\xi}). \tag{13}$$

4. Estimation and Simulation

Here, we implement a well-known estimation approach to obtain the maximum likelihood estimators (MLEs) of $(\alpha,\boldsymbol{\xi})$ expressed by $\left(\hat{\alpha}_{MLE}, \hat{\boldsymbol{\xi}}_{MLE}\right)$. After obtaining $\hat{\alpha}_{MLE}$ and $\hat{\boldsymbol{\xi}}_{MLE}$, a simulation is conducted to see their behaviors/performances.

4.1. Estimation

Consider a random sample, for example, $X_1, X_2, \ldots, X_n$ of size n taken from $f(x;\alpha,\boldsymbol{\xi})$ with parameters α and $\boldsymbol{\xi}$. Then, corresponding to $f(x;\alpha,\boldsymbol{\xi})$, the likelihood function (LF), denoted by $\lambda(\alpha,\theta,\gamma|x_1,x_2,\ldots,x_n)$, is given by

$$\lambda(\alpha,\theta,\gamma|x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} f(x_i;\alpha,\boldsymbol{\xi}). \tag{14}$$

Using Equation (5) in Equation (14), we obtain

$$\lambda(\alpha,\theta,\gamma|x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} \frac{2\alpha\theta\gamma x_i^{\theta-1} e^{-\gamma x_i^\theta}\left(1-e^{-\gamma x_i^\theta}\right)}{(e^\alpha - 1)} e^{\alpha\left(1-\left[1-e^{-\gamma x_i^\theta}\right]^2\right)}. \tag{15}$$

In link to Equation (15), the log LF $\ell(\alpha,\theta,\gamma)$ is given by

$$\begin{aligned}\ell(\alpha,\theta,\gamma) &= n\log 2 + n\log\alpha + n\log\theta + n\log\gamma + (\theta-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\gamma x_i^\theta \\ &\quad + \sum_{i=1}^{n}\log\left(1-e^{-\gamma x_i^\theta}\right) + \sum_{i=1}^{n}\alpha\left(1-\left[1-e^{-\gamma x_i^\theta}\right]^2\right) - n\log(e^\alpha - 1).\end{aligned} \tag{16}$$

Corresponding to Equation (16), the partial derivatives based on α, θ, and γ are given by

$$\frac{\partial}{\partial\alpha}\ell(\alpha,\theta,\gamma) = \frac{n}{\alpha} + n - \sum_{i=1}^{n}\left(1-e^{-\gamma x_i^\theta}\right)^2 - \frac{ne^\alpha}{e^\alpha - 1},$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\ell(\alpha,\theta,\gamma) &= \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i - \gamma\sum_{i=1}^{n}(\log x_i)x_i^\theta + \gamma\sum_{i=1}^{n}\frac{(\log x_i)x_i^\theta e^{-x_i^\theta\gamma}}{\left(1-e^{-\gamma x_i^\theta}\right)} \\ &\quad - 2\alpha\gamma\sum_{i=1}^{n}(\log x_i)x_i^\theta e^{-\gamma x_i^\theta}\left(1-e^{-\gamma x_i^\theta}\right),\end{aligned}$$

and

$$\frac{\partial}{\partial\gamma}\ell(\alpha,\theta,\gamma) = \frac{n}{\gamma} - \sum_{i=1}^{n}x_i^\theta + \sum_{i=1}^{n}\frac{x_i^\theta e^{-\gamma x_i^\theta}}{\left(1-e^{-\gamma x_i^\theta}\right)} - 2\alpha\sum_{i=1}^{n}x_i^\theta e^{-\gamma x_i^\theta}\left(1-e^{-\gamma x_i^\theta}\right),$$

respectively.

On solving $\frac{\partial}{\partial\alpha}\ell(\alpha,\theta,\gamma) = 0$, $\frac{\partial}{\partial\theta}\ell(\alpha,\theta,\gamma) = 0$, and $\frac{\partial}{\partial\gamma}\ell(\alpha,\theta,\gamma) = 0$, we obtain $\hat{\alpha}_{MLE}$ $\hat{\theta}_{MLE}$, and $\hat{\gamma}_{MLE}$, respectively. As we can see, the expressions of the MLEs are not in explicit forms. Therefore, we need to use an iterative procedure such as the Newton–Raphson method to obtain the estimates of the parameters numerically.

In the practice of statistical applications, the asymptotic variance–covariance matrix is an important factor. It provides useful information about the precision and uncertainty of the MLEs. The variance–covariance matrix is constructed with the help of an information matrix whose elements are obtained using the second-order derivatives of the log-likelihood functions of the MLEs. The elements of the information matrix are obtained by taking the negative expectation of the second-order derivatives of the log-likelihood functions. In the present three-parameter case, the variance–covariance matrix is given by

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}^{-1} = \begin{bmatrix} V(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta}) & Cov(\hat{\alpha},\hat{\gamma}) \\ Cov(\hat{\theta},\hat{\alpha}) & V(\hat{\theta}) & Cov(\hat{\theta},\hat{\gamma}) \\ Cov(\hat{\gamma},\hat{\alpha}) & Cov(\hat{\gamma},\hat{\theta}) & V(\hat{\gamma}) \end{bmatrix}^{-1},$$

where

$$I_{11} = \frac{\partial}{\partial \alpha^2} \ell(\alpha,\theta,\gamma) = \frac{ne^{2\alpha}}{(e^{\alpha}-1)^2} - \frac{ne^{\alpha}}{e^{\alpha}-1} - \frac{n}{\alpha^2},$$

$$\begin{aligned}
I_{22} = \frac{\partial}{\partial \theta^2} \ell(\alpha,\theta,\gamma) = &-\gamma \sum_{i=1}^{n} x_i^{\theta} (\log x_i)^2 + 2\alpha\gamma^2 \sum_{i=1}^{n} x_i^{2\theta} (\log x_i)^2 e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right) \\
&- \gamma^2 \sum_{i=1}^{n} \frac{x_i^{2\theta} (\log x_i)^2 e^{-\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)} + \gamma \sum_{i=1}^{n} \frac{x_i^{\theta} (\log x_i)^2 e^{-\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)} \\
&- 2\alpha\gamma \sum_{i=1}^{n} x_i^{\theta} (\log x_i)^2 e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right) - \frac{n}{\theta^2} \\
&- 2\alpha\gamma^2 \sum_{i=1}^{n} x_i^{2\theta} (\log x_i) e^{-2\gamma x_i^{\theta}} \\
&- \gamma^2 \sum_{i=1}^{n} \frac{x_i^{2\theta} (\log x_i)^2 e^{-\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)^2},
\end{aligned}$$

$$\begin{aligned}
I_{33} = \frac{\partial}{\partial \gamma^2} \ell(\alpha,\theta,\gamma) = &\, 2\alpha \sum_{i=1}^{n} x_i^{2\theta} e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right) - \sum_{i=1}^{n} \frac{x_i^{2\theta} e^{-\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)} \\
&- \sum_{i=1}^{n} \frac{x_i^{2\theta} e^{-2\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)^2} - 2\alpha \sum_{i=1}^{n} x_i^{2\theta} e^{-2\gamma x_i^{\theta}} - \frac{n}{\gamma^2},
\end{aligned}$$

$$I_{12} = \frac{\partial}{\partial \alpha \partial \theta} \ell(\alpha,\theta,\gamma) = -2\gamma \sum_{i=1}^{n} x_i^{\theta} (\log x_i) e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right),$$

$$I_{13} = \frac{\partial}{\partial \alpha \partial \gamma} \ell(\alpha,\theta,\gamma) = -2 \sum_{i=1}^{n} x_i^{\theta} e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right),$$

and

$$\begin{aligned}
I_{32} = \frac{\partial}{\partial \gamma \partial \theta} \ell(\alpha,\theta,\gamma) = &\, 2\alpha\gamma \sum_{i=1}^{n} x_i^{2\theta} (\log x_i) e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right) - \sum_{i=1}^{n} x_i^{\theta} (\log x_i) \\
&- \gamma \sum_{i=1}^{n} \frac{x_i^{2\theta} (\log x_i) e^{-\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)} + \sum_{i=1}^{n} \frac{x_i^{\theta} (\log x_i) e^{-\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)} \\
&- 2\alpha \sum_{i=1}^{n} x_i^{\theta} (\log x_i) e^{-\gamma x_i^{\theta}} \left(1 - e^{-\gamma x_i^{\theta}}\right) \\
&- 2\alpha\gamma \sum_{i=1}^{n} x_i^{2\theta} (\log x_i) e^{-2\gamma x_i^{\theta}} \\
&- \gamma \sum_{i=1}^{n} \frac{x_i^{2\theta} (\log x_i) e^{-2\gamma x_i^{\theta}}}{\left(1 - e^{-\gamma x_i^{\theta}}\right)^2}.
\end{aligned}$$

4.2. Simulation

Here, we assess the performances of $\hat{\alpha}_{MLE}$ and $\hat{\boldsymbol{\xi}}_{MLE}$, by incorporating a brief simulation study. To carry out this study, we implement the inverse DF method to generate the random numbers from the E-Weibull distribution with DF $F(x;\alpha,\boldsymbol{\xi})$ and PDF $f(x;\alpha,\boldsymbol{\xi})$ presented in Equation (4) and Equation (5), respectively.

It is important to note that we can perform the simulation study using the initial values of the parameters within the given range of the parameters. There are no hard and fast rules over selecting the initial values of the parameters. Within the given range of the parameters, we can choose any value. In this subsection, we perform a simulation study for two different combination sets of the model parameters, such as (*a*) $\theta = 0.6, \gamma = 1.2, \alpha = 0.8$, and (*b*) $\theta = 1.2, \gamma = 0.5, \alpha = 1$.

For both two sets of parameters presented in (*a*) and (*b*), a random sample of sizes $n = 50, 100, 150, \ldots, 1000$ are generated by implementing the following formula:

$$Q_u = G^{-1}\left(1 - \frac{\log(e^{\alpha} - u[e^{\alpha} - 1])}{\alpha}\right)^{\frac{1}{2}}.$$

The simulation results are replicated 1000 times. To evaluate the performances of $\hat{\alpha}_{MLE}$ and $\hat{\boldsymbol{\xi}}_{MLE}$, two statistical approaches/procedures are considered. The analytical results of these quantities are, respectively, obtained as

$$Bias\left(\hat{\boldsymbol{\Theta}}\right) = \frac{1}{1000}\sum_{k=1}^{1000}\left(\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\right),$$

and

$$MSE\left(\hat{\boldsymbol{\Theta}}\right) = \frac{1}{1000}\sum_{k=1}^{1000}\left(\hat{\boldsymbol{\Theta}}_k - \boldsymbol{\Theta}\right)^2,$$

where $\boldsymbol{\Theta} = (\alpha, \theta, \gamma)$.

The simulation results are obtained by implementing the R-script with the L-BFGS-B algorithm. For more information about the L-BFGS-B algorithm, we refer to [22]. Corresponding to (*a*) $\theta = 0.6, \gamma = 1.2, \alpha = 0.8$, the simulation result of the E-Weibull distribution is presented in Table 1 and displayed visually in Figure 5, whereas in relation to (*b*) $\theta = 1.2, \gamma = 0.5, \alpha = 1$, the simulation result of the E-Weibull distribution is presented in Table 2 and displayed visually in Figure 6.

Table 1. Simulation results for the E-Weibull distribution.

Set 1: $\theta = 0.6, \gamma = 1.2, \alpha = 0.8$.				
n	**Parameters**	**Estimates**	**MSEs**	**Biases**
50	θ	0.6049476	0.00499659	0.00494755
	γ	1.2551280	0.14450400	0.05512813
	α	1.0426544	0.59331445	0.24265444
100	θ	0.6005533	0.00234897	0.00055333
	γ	1.2425210	0.08386770	0.04252140
	α	0.9534761	0.24032552	0.15347606
150	θ	0.6002110	0.00153915	0.00021101
	γ	1.2112220	0.06873700	0.01122178
	α	0.8766179	0.11933972	0.07661793
200	θ	0.6020050	0.00105163	0.00200500
	γ	1.2165500	0.04596168	0.01654977
	α	0.8515907	0.06649227	0.05159065
300	θ	0.6006022	0.00077746	0.00060218
	γ	1.2058170	0.03372750	0.00581712
	α	0.8327267	0.04568434	0.03272672
400	θ	0.6024499	0.00055874	0.00244992
	γ	1.2042760	0.02571049	0.00427639
	α	0.8217224	0.02968523	0.02172245
500	θ	0.5998417	0.00049501	−0.00015833
	γ	1.2007200	0.01915068	0.00071994
	α	0.8191257	0.02544670	0.01912566

Table 1. *Cont.*

Set 1: $\theta = 0.6, \gamma = 1.2, \alpha = 0.8$.				
n	Parameters	Estimates	MSEs	Biases
600	θ	0.6003340	0.00036804	0.00033401
	γ	1.1980350	0.01698582	−0.00196531
	α	0.8134189	0.01992697	0.01341893
700	θ	0.6004238	0.00032848	0.00042383
	γ	1.2009870	0.01392998	0.00098655
	α	0.8148440	0.01676339	0.01484398
800	θ	0.5997210	0.00027106	−0.00027895
	γ	1.2043020	0.01180620	0.00430183
	α	0.8134790	0.01452197	0.01347903
900	θ	0.6003043	0.00024637	0.00030430
	γ	1.1964130	0.01043127	−0.00358707
	α	0.8086972	0.01221243	0.00869715
1000	θ	0.6003874	0.00022317	0.00038743
	γ	1.1995220	0.00906631	−0.00047760
	α	0.8088841	0.01085164	0.00888406

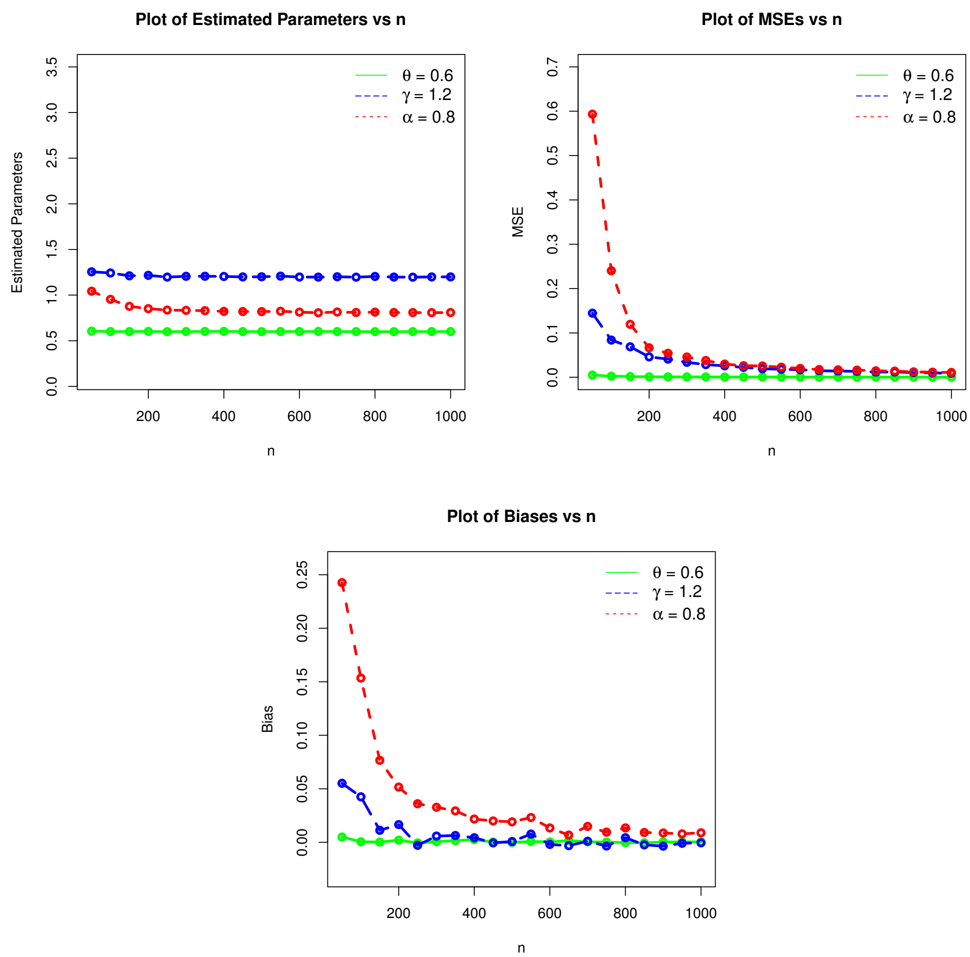

Figure 5. A visual display of the simulation results of the E-Weibull distribution for $\theta = 0.6$, $\gamma = 1.2, \alpha = 0.8$.

Table 2. Simulation results for the E-Weibull distribution.

Set 2: $\theta = 1.2, \gamma = 0.5, \alpha = 1$				
n	Parameters	Estimates	MSEs	Biases
50	θ	1.2259760	0.03178910	0.02597563
	γ	0.5115946	0.02014171	0.01159464
	α	1.2608530	0.78577385	0.26085267
100	θ	1.2106610	0.01362797	0.01066061
	γ	0.5046863	0.01113516	0.00468632
	α	1.1380080	0.32288255	0.13800823
150	θ	1.2053130	0.00853220	0.00531329
	γ	0.5065875	0.00677564	0.00658753
	α	1.0834850	0.14853435	0.08348485
200	θ	1.2037680	0.00642700	0.00376764
	γ	0.4990183	0.00528436	−0.00098173
	α	1.0508120	0.09908880	0.05081233
300	θ	1.2014800	0.00439835	0.00148045
	γ	0.5036268	0.00315291	0.00362679
	α	1.0374950	0.05371933	0.03749508
400	θ	1.2003880	0.00325330	0.00038808
	γ	0.5023467	0.00249109	0.00234669
	α	1.0347320	0.04488695	0.03473155
500	θ	1.2027780	0.00258701	0.00277782
	γ	0.5020358	0.00214560	0.00203577
	α	1.0199860	0.03393216	0.01998558
600	θ	1.2008270	0.00223004	0.00082663
	γ	0.5012683	0.00155480	0.00126833
	α	1.0210830	0.02414743	0.02108303
700	θ	1.2004010	0.00175688	0.00040114
	γ	0.5002717	0.00138508	0.00027174
	α	1.0141590	0.02154580	0.01415942
800	θ	1.2021160	0.00167774	0.00211633
	γ	0.5009623	0.00120435	0.00096234
	α	1.0130060	0.01881078	0.01300580
900	θ	1.2003070	0.00141695	0.00030684
	γ	0.4993483	0.00100807	−0.00065170
	α	1.0051790	0.01494259	0.00517902
1000	θ	1.2019830	0.00137085	0.00198250
	γ	0.5003773	0.00104251	0.00037729
	α	1.0105540	0.01719370	0.01055399

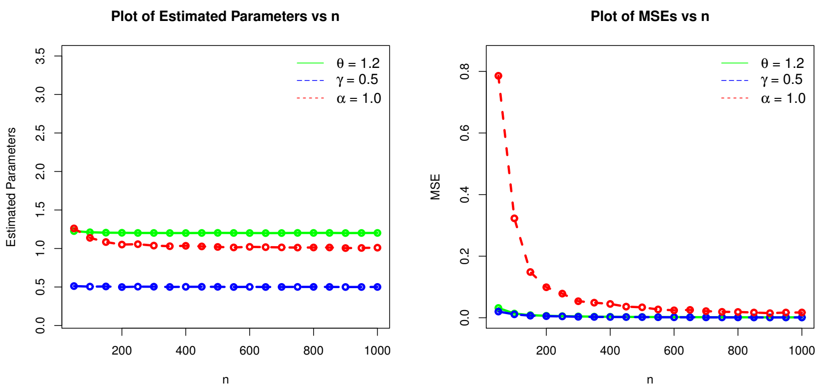

Figure 6. *Cont.*

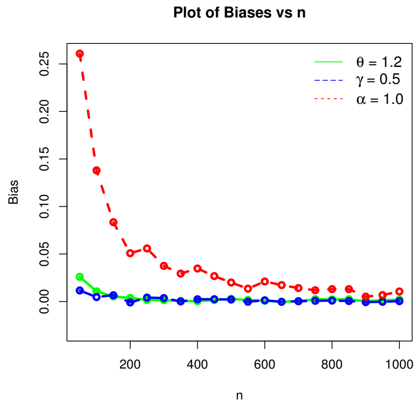

Figure 6. A visual display of the simulation results of the E-Weibull distribution for $\theta = 1.2, \gamma = 0.5, \alpha = 1$.

5. Applications

Here, we implement the E-Weibull distribution to two data sets taken from the field of civil engineering. Both the data sets represent the fracture toughness of the alumina (Al_2O_3) material. The data sets are measured in the units of MPa $m^{\underline{1/2}}$. Using certain evaluation criteria, we compare the performance (i.e., fitting power) of the E-Weibull distribution with other competing distributions.

5.1. Descriptions of the Data Sets

The first data set represents the fracture toughness of Al_2O_3 and is taken from [23]. Onward, we call the first data set Data 1. The observations of Data 1 are given by 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4,5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

The second data set also represents the fracture toughness of Al_2O_3 and is taken from https://data.world/datasets/aluminum (accessed on 12 July 2023). Onward, we call the second data set Data 2. The observations of Data 2 are given by 7.060066, 6.418242, 6.289877, 8.215349, 6.546606, 6.674971, 6.674971, 6.418242, 6.033147, 5.134593, 5.776417, 5.391323, 5.262958, 5.853436, 6.431078, 6.033147, 4.017819, 4.004983, 3.440177, 3.555706, 3.465850, 3.029410, 5.622380, 7.355305, 5.583870, 8.741645, 2.451768, 3.414504, 3.350322, 2.156529, 2.618643, 2.669988, 2.734171, 4.877864, 4.788008, 4.762335, 4.210366, 5.006228, 5.134593, 4.877864, 5.262958, 5.006228, 5.198776, 5.134593, 5.070411, 5.134593, 5.776417, 5.776417, 5.391323, 5.840600, 5.968965, 5.262958, 5.455505, 5.519688, 5.776417, 6.033147, 6.610789, 5.519688, 5.776417, 6.289877, 6.418242, 6.867518, 6.610789, 6.739154, 7.445160, 7.509343, 7.573525, 7.380978, 8.022802, 7.766072, 7.573525, 4.621134, 5.262958, 5.776417, 6.803336, 6.225694, 6.803336, 6.995883, 6.546606, 6.803336, 6.674971, 6.803336, 6.739154, 6.097329, 5.776417, 5.391323, 5.134593, 5.327140, 5.455505, 5.519688, 4.813681, 5.070411, 4.505606, 5.301467, 6.931701, 6.418242, 2.695661, 5.904782, 4.107675, 3.209121, 5.262958, 4.492769, 4.107675, 4.236039, 5.904782, 5.519688, 5.519688, 5.776417, 7.060066, 5.904782, 6.289877, 5.519688, 3.850945, 4.364404, 4.749499, 5.648053, 6.289877, 6.289877, 6.418242, 10.654281, 9.370633, 7.188430, 8.728808, 7.958619, 9.884092, 9.370633, 10.525916, 10.140822, 8.343714, 8.600444.

Corresponding to Data 1 and Data 2, related to the fracture toughness of Al_2O_3, the histograms, total time on test plot transformation (TTT-transform), box plots, and violin plots are presented in Figure 7 and Figure 8, respectively. For more detailed information about the TTT-transform and violin plot, we refer to [24,25].

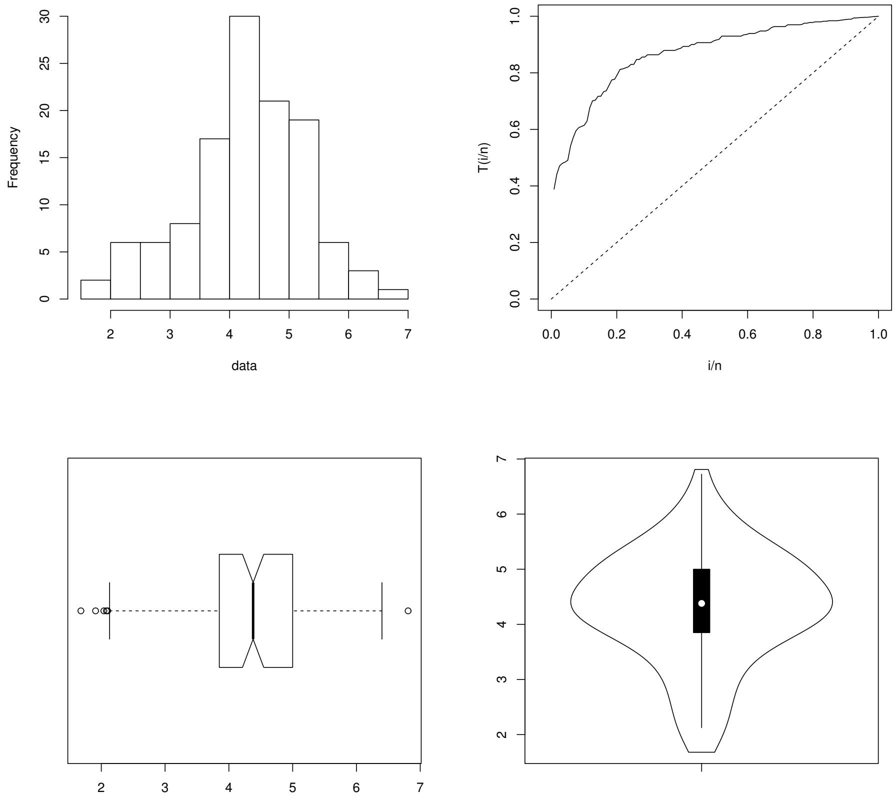

Figure 7. The histogram, TTT-transform, box plot, and violin plot using Data 1.

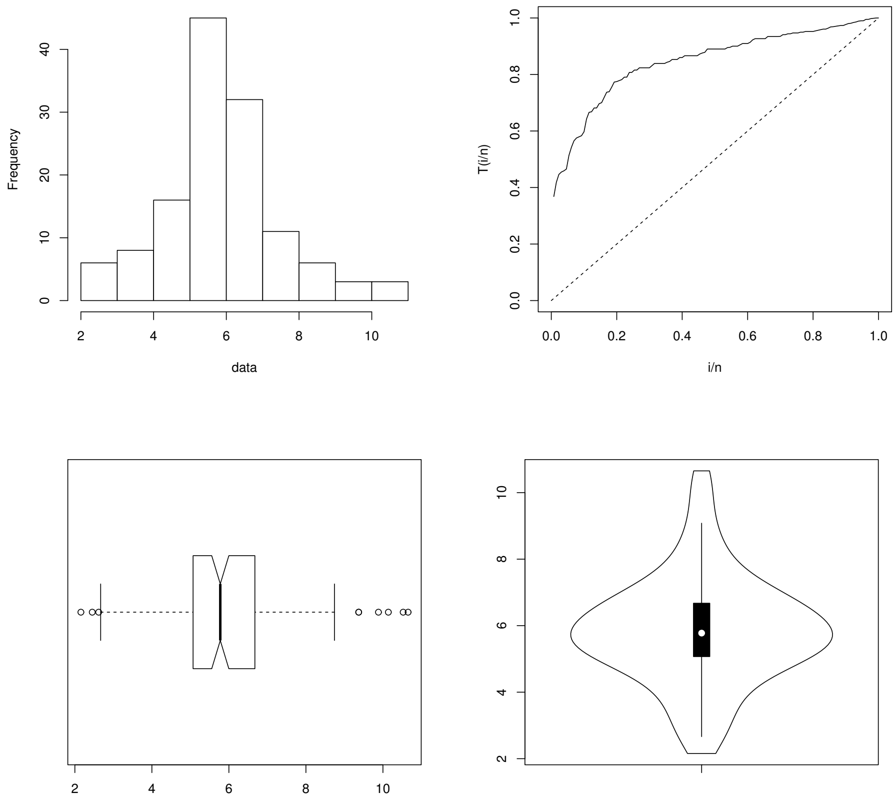

Figure 8. The histogram, TTT-transform, box plot, and violin plot using Data 2.

5.2. The Rival Distributions

To demonstrate the utility and superiority of the E-Weibull distribution over other distributions, we consider some competing/rival distributions. The rival distributions are the prominent and well-known modifications of the Weibull distribution. The DFs of the rival distributions are

- The exponentiated Weibull (Exp-Weibull) distribution of Mudholkar and Srivastava [26] with DF as follows:

$$F(x;\delta_1,\gamma,\theta) = \left(1 - e^{-\gamma x^{\theta}}\right)^{\delta_1}, \qquad x \in \mathbb{R}^+, \delta_1, \gamma, \theta \in \mathbb{R}^+.$$

- The Kumaraswamy Weibull (Kum-Weibull) distribution of Cordeiro et al. [27] with DF as follows:

$$F(x;\delta_1,\delta_2,\gamma,\theta) = 1 - \left(1 - \left[1 - e^{-\gamma x^{\theta}}\right]^{\delta_1}\right)^{\delta_2}, \qquad x \in \mathbb{R}^+, \delta_1, \delta_2, \gamma, \theta \in \mathbb{R}^+.$$

- A New Alpha Power Cosine-Weibull (NAC-Weibull) of Alghamdi and Abd El-Raouf [28] with DF as follows:

$$F(x;\theta,\gamma,\alpha_1) = \frac{\alpha_1^{\cos\left(\frac{\pi}{2} - \frac{\pi\left(1-e^{-\gamma x^{\theta}}\right)}{2}\right)} - 1}{\alpha_1 - 1}, \qquad x \in \mathbb{R}^+, \theta, \gamma, \alpha_1 \in \mathbb{R}^+, \alpha_1 \neq 1.$$

- The exponentiated Flexible Weibull (EF-Weibull) of El-Gohary et al. [29] with DF as follows:

$$F(x;\gamma,\beta,\alpha_1) = \left(1 - e^{-e^{\left(\gamma x - \frac{\beta}{x}\right)}}\right)^{\delta_1} \qquad x \in \mathbb{R}^+, \gamma, \beta, \delta_1 \in \mathbb{R}^+.$$

5.3. The Evaluation Criteria

After selecting the competing models, next we consider seven statistical measures with p-values to see which model provides the closest fit to the fracture toughness of Al_2O_3 data sets. The numerical values of these statistical measures are computed as

- The Akaike information criteria (AIC)

$$AIC = 2p - 2\ell(\mathbf{\Theta});$$

- The Bayesian information criteria (BIC)

$$BIC = p\log(m) - 2\ell(\mathbf{\Theta});$$

- The Consistent Akaike information criteria (CAIC)

$$CAIC = \frac{2mp}{m-p-1} - 2\ell(\mathbf{\Theta});$$

- The Hannan–Quinn information criteria (HQIC)

$$HQIC = 2p\log(\log(m)) - 2\ell(\mathbf{\Theta});$$

- The Anderson–Darling (AD) test

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_i) + \log\{1 - F(x_{n-i+1})\}];$$

- The Cramer–von Mises (CM) test

$$CM = \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{2i-1}{2n} - F(x_i) \right]^2;$$

- The Kolmogorov–Smirnov (KS) test

$$KS = sup_x \left| F_n(x) - F(x) \right|.$$

The values of the MLEs along with their corresponding confidence intervals and the above decisive measures are calculated using the package AdequacyModel in R-script with optim() and method = BFGS.

A model having the lowest values of the above statistical measures indicates a close fit to the data. Based on these measures, it is observed that the E-Weibull distribution has smaller values of these analytical measures as compared to the Exp-Weibull and Kum-Weibull distributions.

5.4. Analysis of Data 1

Corresponding to the first data set related to the fracture toughness of Al_2O_3, the values of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{\delta}_{1MLE}$, $\hat{\delta}_{2MLE}$, $\hat{\alpha}_{1MLE}$, and $\hat{\beta}_{MLE}$ are provided in Table 3. Furthermore, the 95 % confidence interval (CI) of the MLEs is also provided in Table 3. The numerical values of the respective statistical measures for the E-Weibull, Exp-Weibull, and Kum-Weibull distributions are obtained in Tables 4 and 5. The analytical results in Tables 4 and 5, confirm the best-fitting capability of the E-Weibull distribution as it has the lowest values of the considered tests. Furthermore, for Data 1, the visual illustration of the performances of the E-Weibull distribution is also considered. For the visual performances, we consider the plots of the estimated PDF, empirical CDF, Kaplan–Meier survival plot, probability–probability (PP), and quantile–quantile (QQ) plots; see Figure 9. From the visual illustrations in Figure 9, it is obvious that the E-Weibull distribution fits Data 1 closely.

Table 3. The values of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{\delta}_{1MLE}$, $\hat{\delta}_{2MLE}$, $\hat{\alpha}_{1MLE}$, and $\hat{\beta}_{MLE}$ for the first data set of Al_2O_3.

Model	$\hat{\alpha}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{\delta}_{1MLE}$	$\hat{\delta}_{2MLE}$	$\hat{\alpha}_{1MLE}$	$\hat{\beta}_{MLE}$
E-Weibull	2.185	2.807	0.030	-	-	-	-
	(4.016, 0.354)	(3.426, 2.188)	(0.066, 0.005)				
Exp-Weibull	-	3.537	0.006	1.737	-	-	-
		(3.883, 3.191)	(0.010, 0.002)	(2.230, 1.244)			
Kum-Weibull	-	2.251	0.013	2.631	14.190	-	-
		(2.996, 1.506)	(0.028, 0.001)	(3.560, 1.704)	(45.340, 1.902)		
NAC-Weibull	-	3.459	0.003	-	-	3.692	-
		(3.659, 3.259)	(0.004, 0.002)			(6.843, 0.541)	
EF-Weibull	-	-	0.300	35.425	-	-	0.326
			(0.319, 0.282)	(66.037, 4.813)			(1.110, 0.016)

Table 4. The values of the AIC, CAIC, BIC, and HQIC of the fitted models using the first data set of Al_2O_3.

Model	**AIC**	**CAIC**	**BIC**	**HQIC**
E-Weibull	344.0993	344.3080	352.4366	347.4848
Exp-Weibull	346.9896	347.1983	355.3270	350.3751
Kum-Weibull	347.3122	347.6631	358.4287	351.8263
NAC-Weibull	346.9654	346.1741	354.3028	349.3509
EF-Weibull	349.4578	349.6665	357.7952	352.8434

Table 5. The values of the CM, AD, KS, and *p*-value of the fitted models using the first data set of Al_2O_3.

Model	CM	AD	KS	*p*-Value
E-Weibull	0.0984	0.6352	0.0644	0.7054
Exp-Weibull	0.1573	0.9780	0.0939	0.2443
Kum-Weibull	0.1261	0.7864	0.0800	0.4315
NAC-Weibull	0.0922	0.7550	0.0955	0.2277
EF-Weibull	0.1631	1.0422	0.0942	0.2408

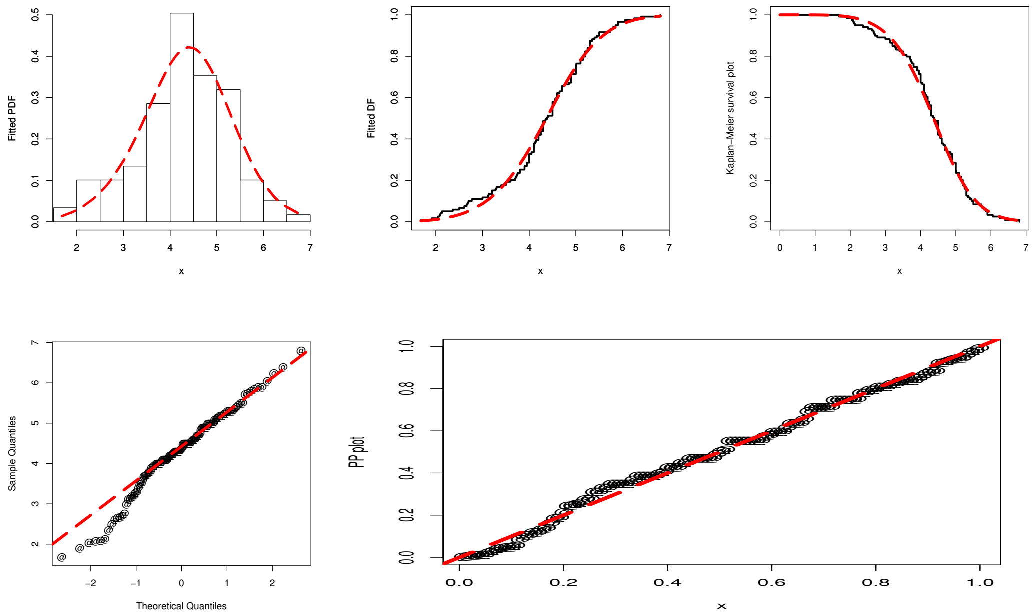

Figure 9. In relation to Data 1, the Fitted PDF, DF, SF, QQ, and PP plots of the E-Weibull distribution.

5.5. Analysis of Data 2

For the second data set of the fracture toughness of Al_2O_3, the values of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{\delta}_{1MLE}$, $\hat{\delta}_{2MLE}$, $\hat{\alpha}_{1MLE}$, and $\hat{\beta}_{MLE}$ along with the 95% CI of the MLEs are reported in Table 6, whereas the values of the statistical tests for the E-Weibull, Exp-Weibull, and Kum-Weibull distributions are presented in Tables 7 and 8. From the numerical description provided in Tables 7 and 8, it is clear that the E-Weibull distribution is the best competitor model. In addition to the numerical results (Tables 7 and 8), the performances of the E-Weibull distribution are also presented visually in Figure 10. The plots in Figure 10 again visually confirm that the E-Weibull distribution closely follows Data 2.

Table 6. The values of $\hat{\alpha}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\gamma}_{MLE}$, $\hat{\delta}_{1MLE}$, $\hat{\delta}_{2MLE}$, $\hat{\alpha}_{1MLE}$, and $\hat{\beta}_{MLE}$ for the second data set of Al_2O_3.

Model	$\hat{\alpha}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\gamma}_{MLE}$	$\hat{\delta}_{1MLE}$	$\hat{\delta}_{2MLE}$	$\hat{\alpha}_{1MLE}$	$\hat{\beta}_{MLE}$
E-Weibull	3.928	1.890	0.086	-	-	-	-
	(5.140, 0.717)	(2.503, 1.276)	(0.208, 0.034)				
Exp-Weibull	-	2.349	0.024	2.813	-	-	-
		(4.596, 1.030)	(0.062, 0.013)	(3.009, 1.688)			
Kum-Weibull	-	1.408	0.095	4.975	4.585	-	-
		(2.151, 1.284)	(0.141, 0.048)	(6.622, 3.327)	(7.953, 1.216)		
NAC-Weibull	-	0.002	2.933	-	-	3.101	-
		(0.003, 0.001)	(4.933, 0.933)			(3.229, 2.973)	
EF-Weibull	-	-	0.208	10.721	-	-	1.104
			(0.224, 0.192)	(26.212, 4.769)			(4.091, 0.083)

Table 7. The values of the AIC, CAIC, BIC, and HQIC of the fitted models using the second data set of Al_2O_3.

Model	AIC	CAIC	BIC	HQIC
E-Weibull	497.2910	497.4815	505.8936	500.7865
Exp-Weibull	499.7503	499.9408	508.3529	503.2459
Kum-Weibull	501.4224	501.7424	512.8926	506.0831
NAC-Weibull	507.0984	507.2889	515.7010	510.5940
EF-Weibull	500.2151	500.4056	508.8177	503.7106

Table 8. The values of the CM, AD, KS, and p-value of the fitted models using the second data set of Al_2O_3.

Model	CM	AD	KS	p-Value
E-Weibull	0.1134	0.6960	0.0666	0.6114
Exp-Weibull	0.1824	1.0656	0.0870	0.2783
Kum-Weibull	0.1740	1.0186	0.0871	0.2771
NAC-Weibull	0.2320	1.3915	0.1134	0.0702
EF-Weibull	0.1919	1.1185	0.0826	0.3376

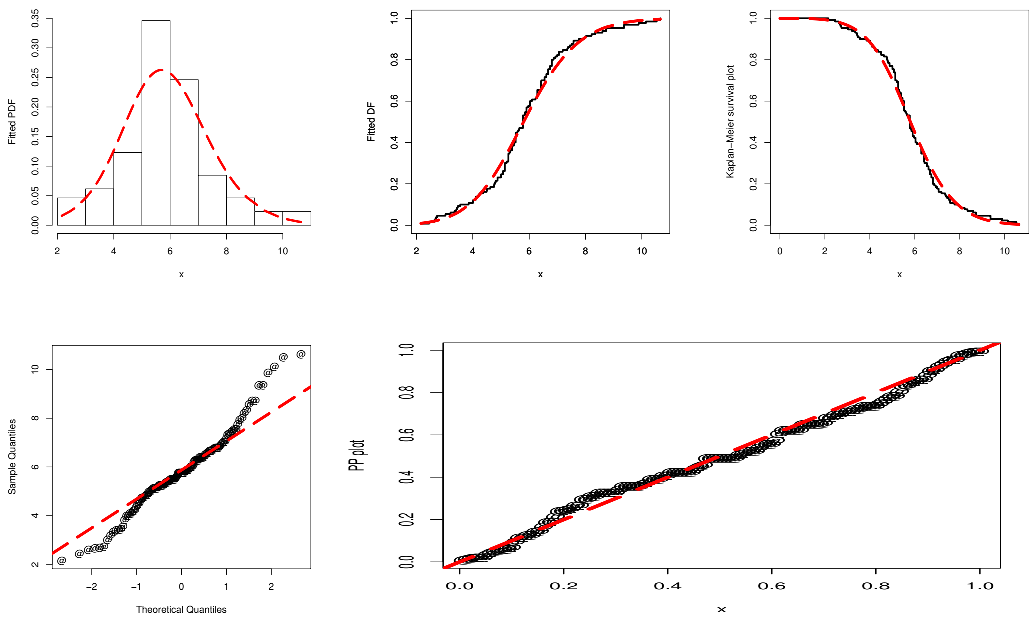

Figure 10. In relation to Data 2, the Fitted PDF, DF, SF, QQ, and PP plots of the E-Weibull distribution.

6. Future Research Work

We highlighted earlier the importance and applications of statistical methodologies in applied fields. In the future, therefore, we are motivated to further extend our model to update its distributional flexibility. Some possible extensions of the E-X distributions could be handled as follows:

- The exponentiated version of the E-X distributions:
  Mudholkar and Srivastava [26] suggested a useful method for extending the existing distributions called the exponentiated family of distributions. The CDF $M(x;\delta_1,\boldsymbol{\Theta})$ of the exponentiated family of distributions is expressed by

$$M(x;\delta_1,\boldsymbol{\Theta}) = (F(x;\boldsymbol{\Theta}))^{\delta_1}, \qquad x \in \mathbb{R}, \delta_1 > 0. \tag{17}$$

In the future, we are intending to study the exponentiated version of the E-X distributions called the exponentiated E-X (EE-X) distributions. The CDF $M(x;\delta_1,\boldsymbol{\Theta})$ of the EE-X distributions is obtained by using Equation (1) in Equation (17), as given by

$$M(x;\delta_1,\boldsymbol{\Theta}) = \left(\frac{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}}{e^{\alpha}-1}\right)^{\delta_1}, \qquad x \in \mathbb{R},$$

where $\boldsymbol{\Theta} = (\alpha,\boldsymbol{\xi})$.

- The Kumaraswamy version of the E-X distributions:
  Cordeiro et al. [27] proposed the Kumaraswamy family of distributions. The CDF $M(x;\delta_1,\delta_2,\boldsymbol{\Theta})$ of the Kumaraswamy family of distributions is given by

$$M(x;\delta_1,\delta_2,\boldsymbol{\Theta}) = 1 - \left[1 - (F(x;\boldsymbol{\Theta}))^{\delta_1}\right]^{\delta_2}, \qquad x \in \mathbb{R}, \delta_1 > 0, \delta_2 > 0. \tag{18}$$

  In the future, we are also committed to study the Kumaraswamy version of the E-X distributions called the Kumaraswamy E-X (KE-X) distributions. The CDF $M(x;\delta_1,\delta_2,\boldsymbol{\Theta})$ of the KE-X distributions is obtained by using Equation (1) in Equation (18), as given by

$$M(x;\delta_1,\delta_2,\boldsymbol{\Theta}) = 1 - \left(1 - \left[\frac{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}}{e^{\alpha}-1}\right]^{\delta_1}\right)^{\delta_2}, \qquad x \in \mathbb{R}.$$

- The Marshall–Olkin version of the E-X distributions:
  Marshall and Olkin [30] introduced a very useful distributional method for obtaining new probability distributions with CDF $M(x;\kappa,\boldsymbol{\Theta})$ given by

$$M(x;\kappa,\boldsymbol{\Theta}) = \frac{F(x;\boldsymbol{\Theta})}{\kappa + (1-\kappa)F(x;\boldsymbol{\Theta})} \qquad x \in \mathbb{R}, \kappa \in \mathbb{R}^+. \tag{19}$$

  As a future study, we can also study a new version of the E-X distributions using the given distributional method in Equation (19). The new modified form of the E-X distributions based on Equation (19) may be called the Marshall–Olkin E-X (MOE-X) distributions. The CDF $M(x;\kappa,\boldsymbol{\Theta})$ of the MOE-X distributions is obtained by using Equation (1) in Equation (19), as given by

$$M(x;\kappa,\boldsymbol{\Theta}) = \frac{\left(\frac{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}}{e^{\alpha}-1}\right)}{\kappa + (1-\kappa)\left(\frac{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}}{e^{\alpha}-1}\right)} \qquad x \in \mathbb{R}.$$

- The alpha power transformed version of the E-X distributions:
  Mahdavi and Kundu [31] used the alpha power transformation method and suggested a useful method for generating new probability distributions with CDF $M(x;\alpha_1,\boldsymbol{\Theta})$ given by

$$M(x;\alpha_1,\boldsymbol{\Theta}) = \frac{\alpha_1^{F(x;\boldsymbol{\Theta})} - 1}{\alpha_1 - 1} \qquad x \in \mathbb{R}, \alpha_1, \in \mathbb{R}^+, \alpha_1 \neq 1. \tag{20}$$

  As a future study, we are also planning to introduce the alpha power transformed version of the E-X distributions called the alpha power transformed E-X (APTE-X) distributions. The CDF $M(x;\alpha_1,\boldsymbol{\Theta})$ of the APTE-X distributions is obtained by using Equation (1) in Equation (20), as given by

$$M(x;\alpha_1,\boldsymbol{\Theta}) = \frac{\alpha_1^{\left(\frac{e^{\alpha} - e^{\alpha\left(1-[G(x;\boldsymbol{\xi})]^2\right)}}{e^{\alpha}-1}\right)} - 1}{\alpha_1 - 1}, \qquad x \in \mathbb{R}.$$

7. Final Remarks

Probability distributions have a great role in civil engineering and other connected fields. These models are very crucial for modeling different kinds of data sets. With the help of probability distributions, we can model and predict the performances of different entities. Keeping in view the crucial role of probability distributions in different engineering sectors, this paper considered a useful approach to obtain new probability distributions, namely, the E-X family. Some statistical properties of the E-X distributions including QF, rth moment, and MGF were derived. The MLEs of the E-X distributions were also obtained. Based on the E-X method, an updated version of the Weibull distribution, namely, the E-Weibull distribution was introduced. To illustrate the E-Weibull distribution, two data sets representing the fracture toughness of Al_2O_3 were analyzed. Based on the selected evaluation criteria, it was observed that the E-Weibull distribution was the best-suited model for analyzing the Al_2O_3 data sets.

Our future goals are centered on understanding and exploring the heavy-tailed characteristics of the proposed model. Furthermore, we are determined to implement the proposed distribution for statistical analysis of financial data sets that possess these heavy-tailed characteristics.

Author Contributions: Conceptualization, H.M.A., Z.A. and O.H.O.; methodology, H.M.A., Z.A., O.H.O. and G.S.R.; software, H.M.A., Z.A., O.H.O. and A.A.-A.H.E.-B.; validation, O.H.O. and Z.A.; formal analysis, H.M.A., Z.A., O.H.O. and G.S.R.; investigation, O.H.O. and A.A.-A.H.E.-B.; data curation, Z.A. and G.S.R.; writing—original draft preparation, H.M.A., Z.A., O.H.O., G.S.R. and A.A.-A.H.E.-B.; writing—review and editing, H.M.A., Z.A. and O.H.O.; visualization, H.M.A., Z.A., G.S.R. and A.A.-A.H.E.-B. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number RI-44-0439.

Data Availability Statement: The data sets are provided within the main body of the paper.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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