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A Singular Tempered Sub-Diffusion Fractional Equation with Changing-Sign Perturbation

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Abstract: In this paper, we establish some new results on the existence of positive solutions for a singular tempered sub-diffusion fractional equation involving a changing-sign perturbation and a lower-order sub-diffusion term of the unknown function. By employing multiple transformations, we transform the changing-sign singular perturbation problem to a positive problem, then establish some sufficient conditions for the existence of positive solutions of the problem. The asymptotic properties of solutions are also derived. In deriving the results, we only require that the singular perturbation term satisfies the Carathéodory condition, which means that the disturbance influence is significant and may even achieve negative infinity near some time singular points.

Keywords: singular perturbation; tempered changing-sign fractional equation; Carathéodory conditions; sub-diffusion

MSC: 34A08; 34A25; 47H14

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1. Introduction

Fractional calculus is an important tool to describe the anomalous diffusion phenomena in Brownian motion where a large number of particles transmit at different speeds. In particular, the anomalous transmission includes the superlinear and sublinear growth for the mean squared displacement with time, which produces the anomalous super-diffusion and sub-diffusion phenomenon. Super-diffusion usually involves a fractional diffusion term which possesses a globally nonlocal transport characteristic, i.e., the flux of a scalar relies on the global spatial distribution of the scalar rather than its local spatial gradients, whereas sub-diffusion can be modeled through a time fractional derivative which possesses a non-locally temporal transport property [1]. In other words, the tempered fractional Brownian motion exhibits semi-long range dependence, which falls off like a power law in moderate time but then eventually becomes short-range dependent at long time scales. The non-locally temporal transport property can be described by a tempered fractional derivative operator ${}_0^R\mathbb{D}_t{}^{\gamma,\lambda}x(t)$, which is actually an exponential optimization of the Riemann–Liouville fractional derivative ${}_0^R\mathscr{D}_t{}^{\gamma}$, i.e., we have the following mathematical relation:

$$ {}_0^R\mathbb{D}_t{}^{\gamma,\lambda}u(t) = e^{-\lambda t}\,{}_0^R\mathscr{D}_t{}^{\gamma}(e^{\lambda t}u(t)), \tag{1} $$

where $\gamma, \lambda > 0$ are constants, ${}_0^R\mathscr{D}_x{}^{\gamma}$ represents the standard Riemann–Liouville fractional derivative defined by

$$ {}_0^R\mathscr{D}_x{}^{\gamma}u(x) = \frac{d^n}{dx^n}\left(I^{n-\gamma}u(x)\right) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \int_0^x (x-y)^{n-\gamma-1}u(y)dy, $$

and

$$I^{\gamma}u(x)=\frac{1}{\Gamma(\gamma)}\int_0^x (x-y)^{\gamma-1}u(y)dy$$

is the Riemann–Liouville fractional integral operator. For more details of the definitions and applications, we refer the reader to [2–6].

From a mathematical point of view, an anomalous diffusion process is the continuous limit of a random walk in a time continuum governed by a power-law probability second moment divergent distribution. However the moments divergent of Lévy processes in bounded domains or with finite time perform the exponential rule rather than the power-law [1]. Thus, the most effective strategy is to multiply an exponential factor in the fractional diffusion term to transform it as a tempered-stable Lévy processes, and then the corresponding continuous limit results in a tempered diffusion fractional model with convergent moments in space [7] or in both space and time [8]. This extension provides a time domain stochastic process model for the famous Davenport spectrum of wind speed [9,10] for designing electric power generation facilities and for studying geophysics problems [11] and finance problems [12]. In recent work [13], Mali et al. developed some theories of tempered fractional calculus, and some properties and applications were also given. Ortigueira and Bengochea [14] introduced the bilateral tempered fractional derivatives to unify the one-sided tempered fractional calculus, the classic, substantial and shifted fractional operators for studying variance gamma processes [15,16] and turbulence model in Statistical Physics [17] and the Regular Lévy Processes of exponential type [18]. Recently, Zhang et al. [19] employed the method of upper and lower solutions to derive some new results for a p-Laplacian singular tempered fractional equation

$$\begin{cases} {}_0^R\mathbb{D}_x{}^{\gamma,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(x)\right)\right)=f(x,u(x)),\\ u(0)=0,\ {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(0)=0,\ u(1)=\int_0^1 e^{-\lambda(1-x)}u(x)dx, \end{cases} \tag{2}$$

with $0<\gamma\le 1, 1<\sigma\le 2$, and the nonlinearity f is decreasing in the second variable. By using the properties of superquadratic and subquadratic functions, Saker et al. [20] established some new refinement multidimensional dynamic inequalities of Hardy's type on time scales. In [21], Zakarya et al. provided novel generalizations by considering the generalized conformable fractional integrals for reverse Copson's type inequalities on time scales. For some other applications of fractional calculus, the reader is referred to [22–36].

This paper deals with the following sub-diffusion model with a changing-sign perturbation.

$$\begin{cases} -{}_0^R\mathbb{D}_x{}^{\gamma,\lambda}u(x)=f\left(x,e^{\lambda x}u(x),{}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(x)\right)-g\left(x,{}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(x)\right),\\ {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(0)=0,\ {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(1)=0, \end{cases} \tag{3}$$

where $1<\gamma\le 2, 0<\sigma<1$ and $\gamma-\sigma>1$, $u(x)$ is particle jump density function, $e^{\lambda x}u(x)$ represents an exponential decay, and ${}_0^R\mathbb{D}_x{}^{\gamma,\lambda}u(x), {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(x)$ are anomalous sub-diffusion terms which are tempered time fractional derivative operators. The main nonlinearity $f:[0,1]\times[0,+\infty)\times(-\infty,+\infty)\to[0,+\infty)$ is continuous, and $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a changing-sign perturbation, which satisfies the Crathèodory condition. The above condition implies that g can be singular at some zero measure set of [0, 1], i.e., g may have infinite many singular points with respect to the time variables, and further, it leads to the global nonlinearity which may tend toward negative infinity.

A changing-sign differential equation with the nonlinear term $f(x,u)\ge -M, M>0$ originated from the study of the chemical reactor theory [37], which is also called the semipositone problem in the literature. Recently, Denk and Topal [38] considered a second-order semipositone m-point BVP on time scales, and by employing the fixed point theorem, it has been proven that the semipositone m-point BVP has triple positive solutions. In [39], by constructing a special cone and combining the properties on time scales, the authors dealt with a third-order semipositone equation on time scales, and the existence of positive solutions was obtained provided that the nonlinear term f can be changing sign. However, there are no results for sub-diffusion model with a changing-sign perturbation even for the perturbation which is only a time-variable function without a lower-order sub-diffusion term. This is mainly because many nonlinear analysis theories and methods, such as the spaces theories [40–44], smooth theories [45–47], operator method [48,49], the method of moving sphere [50], critical point theories [51–54] and iterative techniques [55–57], have not been used to solve the sub-diffusion model when the main nonlinear term f and the changing-sign perturbation g all involve a lower-order tempered fractional sub-diffusion term. In the present paper, by using the spaces theories, regularity theories, operator theories and the technique of moving plane, we firstly transform the changing-sign sub-diffusion model to a positive problem and then derive sufficient conditions on the existence of positive solutions of the changing-sign sub-diffusion model (3) based on the fixed-point theorem in the cone. The new contributions in this paper include the following aspects:

(i) The existence of positive solutions for a sub-diffusion model with a changing-sign perturbation is derived under the cases in which the main nonlinearity f is superlinear or sublinear.
(ii) Only the Carathéodory condition is required for the singular perturbation, which makes the disturbance influence to be significant so that the whole nonlinearity may tend to achieve negative infinity near some time singular points in $[0,1]$.
(iii) The main nonlinear term f and the negative perturbation g all involve a lower-order tempered fractional sub-diffusion term of unknown functions.
(iv) The singular perturbation g is allowed to have infinitely many singular points in $[0,1]$.
(v) The asymptotic properties of positive solutions are studied.

This work is structured as follows. In Section 2, we firstly construct our work space and study the unique positive solution for a tempered linear fractional equation, and then transform the changing-sign sub-diffusion model to a positive problem by the moving plane technique. In Section 3, some sufficient conditions on the existence of positive solutions for the changing-sign sub-diffusion model (3) are derived for the case in which the main nonlinear term f is superlinear or sublinear. Some examples are given to illustrate our main results in Section 4.

2. Preliminaries and Lemmas

Let $E = C([0,1];\mathbb{R})$ be the work space of this paper. Clearly, E is a Banach space which possesses the norm

$$||z|| = \max_{x\in[0,1]} |z(x)|.$$

Define

$$P = \{z \in E : z(x) \geq x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}||z||\},$$

then P is a cone of E.

Definition 1. *We say the map $(x,z) \mapsto g(x,z)$ in $[0,1]\times\mathbb{R}$ satisfies the Cratheódory condition if*

(i) *$x \mapsto g(x,z)$ is Lebesgue measurable for every $z \in \mathbb{R}$;*
(ii) *$z \mapsto g(x,z)$ is continuous for a.e. $x \in [0,1]$;*
(iii) *for a.e. $x \in [0,1]$ and any $z \in \mathbb{R}$, there exists a function $\hbar \in L^1[0,1]$ such that*

$$|g(x,z)| < \hbar(x).$$

Remark 1. *From (ii) and (iii) of Definition 1, $g(x,z)$ can be singular or undefined at some zero measure set of $[0,1]$. For example,*

$$g(x,z) = \frac{e^{-z}+2}{\prod_{i=1}^{n}|x-a_i|^{\frac{1}{2}}} < \frac{4}{\prod_{i=1}^{n}|x-a_i|^{\frac{1}{2}}},\ x\in[0,1],\ z\in[0,+\infty), 0<a_i<1, i=1,2,3...,m.$$

Clearly, $g(x,z)$ is singular and tends to infinity at a zero measure set $\{a_1,a_2,a_3....,a_m\} \subset [0,1]$. This indicates that the disturbance influence of the singular perturbation term is significant and achieves negative infinity near the singular points $\{a_1,a_2,a_3....,a_m\}$.

Remark 2. *In singular points, the loss of continuity of the function will make the corresponding operator lose compactness, which may cause many theories of nonlinear functional analysis falling out of use. In order to overcome this difficulty, we introduce the Cratheódory condition to govern the contribution of the singular perturbation term which makes the corresponding nonlinear operator well-defined under this condition.*

Now, we list the assumptions used in the rest of the paper.

(G1) $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies the Crathèodory condition, and we denote

$$\mu = \frac{1}{\Gamma(\gamma-\sigma)}\int_0^1 \hbar(y)dy.$$

(G2) $f\in C([0,1]\times[0,+\infty)\times(-\infty,+\infty),[0,+\infty))$.

(G3) There exists a subinterval $[a,b]\subset(0,1)$ such that

$$\lim_{u+v\to+\infty}\min_{x\in[a,b]}\frac{f(x,u,v)}{u+v} = +\infty.$$

(G4)

$$\lim_{|u+v|\to\infty}\max_{x\in[0,1]}\frac{f(x,u,v)}{|u+v|} = 0.$$

The following lemmas are necessary.

Lemma 1 ([19]). *Let $\gamma>\sigma>0$ and $z(x)\in C[0,1]\cap L^1[0,1]$. Then,*

(i)

$$I^{\sigma}{}_0^R\mathscr{D}_{\boldsymbol{x}}{}^{\sigma}z(x) = z(x)+c_1x^{\sigma-1}+c_2x^{\sigma-2}+\cdots+c_mx^{\sigma-m}, \tag{4}$$

where $c_i\in\mathbb{R}, i=1,2,3,...,m, (m=[\sigma]+1)$, $[\sigma]$ denotes the greatest integer part of the number σ;

(ii)

$$I^{\gamma}I^{\sigma}z(x) = I^{\gamma+\sigma}z(x),\ \ {}_0^R\mathscr{D}_{\boldsymbol{x}}{}^{\sigma}I^{\gamma}z(x) = I^{\gamma-\sigma}z(x),\ \ {}_0^R\mathscr{D}_{\boldsymbol{x}}{}^{\sigma}I^{\sigma}z(x) = z(x).$$

Lemma 2 ([3,19]). *If $1<\gamma-\sigma\le 2$, then the linear singular tempered fractional equation*

$$\begin{cases} -{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}\ell(x) = \hbar(x), \\ \ell(0)=0,\ \ \ell(1)=0, \end{cases} \tag{5}$$

has a unique positive solution $\ell(x)$ with the form

$$\ell(x) = \int_0^1 G(x,y)\hbar(y)dy, \tag{6}$$

where

$$G(x,y)=\begin{cases}\dfrac{x^{\gamma-\sigma-1}(1-y)^{\gamma-\sigma-1}-(x-y)^{\gamma-\sigma-1}}{\Gamma(\gamma-\sigma)}e^{-\lambda x}e^{\lambda y}, 0\le y\le x\le 1;\\ \dfrac{x^{\gamma-\sigma-1}(1-y)^{\gamma-\sigma-1}}{\Gamma(\gamma-\sigma)}e^{-\lambda x}e^{\lambda y},\ \ 0\le x\le y\le 1\end{cases} \tag{7}$$

is the Green function of (5).

Proof. Since $1<\gamma-\sigma\le 2$, it follows from (1) and (4) that

$$e^{\lambda x}\ell(x)=-\frac{1}{\Gamma(\gamma-\sigma)}\int_0^x (x-y)^{\gamma-\sigma-1}e^{\lambda y}\hbar(y)dy+b_1x^{\gamma-\sigma-1}+b_2x^{\gamma-\sigma-2}, x\in[0,1].$$

Noticing that $\ell(0)=0$ and $\ell(1)=0$, one gets $b_2=0$ and

$$b_1=\frac{1}{\Gamma(\gamma-\sigma)}\int_0^1 (1-y)^{\gamma-\sigma-1}e^{\lambda y}\hbar(y)dy.$$

So

$$\begin{aligned}\ell(x)&=\frac{1}{\Gamma(\gamma-\sigma)}\left[\int_0^1(1-y)^{\gamma-\sigma-1}e^{-\lambda x}e^{\lambda y}\hbar(y)dyx^{\gamma-\sigma-1}-\int_0^x(x-y)^{\gamma-\sigma-1}e^{-\lambda x}e^{\lambda y}\hbar(y)dy\right]\\&=\frac{1}{\Gamma(\gamma-\sigma)}\int_0^x\left[x^{\gamma-\sigma-1}(1-y)^{\gamma-\sigma-1}-(x-y)^{\gamma-\sigma-1}\right]e^{-\lambda x}e^{\lambda y}\hbar(y)dy\\&+\frac{1}{\Gamma(\gamma-\sigma)}\int_x^1 x^{\gamma-\sigma-1}(1-y)^{\gamma-\sigma-1}e^{-\lambda x}e^{\lambda y}\hbar(y)dyx^{\gamma-\sigma-1}\\&=\int_0^1 G(x,y)\hbar(y)dy,\ \ x\in[0,1].\end{aligned}$$

□

Remark 3. *In this paper, the boundary conditions of Equation* (3) *are Dirichlet type, of course, and the boundary conditions of the equation can also be changed to other types of boundary conditions; however, this change affects only the form of the Green function and does not affect the entire proof process. So the boundary conditions are critical but not essential.*

Lemma 3 ([22])**.** *If $1<\gamma\le 2$, $0<\sigma<1$ satisfying $\gamma-\sigma>1$, for Green function $G(x,y)$, we have the following properties:*

(1) For any $(x,y)\in[0,1]\times[0,1]$, $G(x,y)$ is a non-negative and continuous function;
(2) For any $(x,y)\in[0,1]\times[0,1]$,

$$\frac{x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\le G(x,y)\le\frac{x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}}{\Gamma(\gamma-\sigma)}\ or\ \left(\frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\right). \tag{8}$$

Lemma 4. *Let $\ell(x)$ be the solution for the linear problem* (5)*, then we have the following estimate*

$$0\le\ell(x)\le\mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}, x\in[0,1]. \tag{9}$$

Proof. In view of (6) and (8), we have

$$\ell(x)=\int_0^1 G(x,y)\hbar(y)dy\le\frac{x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}}{\Gamma(\gamma-\sigma)}\int_0^1\hbar(y)dy=\mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}, x\in[0,1].$$

□

Now, we make the transformation

$$u(x) = e^{-\lambda x} I^{\sigma}(e^{\lambda x} v(x)), v(x) \in C[0,1],$$

and consider the Dirichlet BVP of the following mixed integro-differential tempered fractional equation

$$\begin{cases} -{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda} v(x) = f(x, I^{\sigma}(e^{\lambda x} v(x)), v(x)) - g(x, v(x)), \\ v(0) = 0, \ \ v(1) = 0. \end{cases} \tag{10}$$

Lemma 5. *Equations (3) and (10) are equivalent. Moreover, if v is a positive solution of Equation (10), then $u(x) = e^{-\lambda x} I^{\sigma}(e^{\lambda x} v(x))$ is a positive solution of Equation (3).*

Proof. Firstly, substitute $u(x) = e^{-\lambda x} I^{\sigma}(e^{\lambda x} v(x))$ into (3). By using (1) and Lemma 1, one has

$$\begin{aligned} {}_0^R\mathbb{D}_x{}^{\gamma,\lambda} u(x)) &= e^{-\lambda x} {}_0^R\boldsymbol{\mathscr{D}_x}{}^{\gamma}(e^{\lambda x} u(x)) \\ &= e^{-\lambda x} \frac{d^n}{dx^n} I^{n-\gamma}(e^{\lambda x} u(x)) \\ &= e^{-\lambda x} \frac{d^n}{dx^n} I^{n-\gamma}(I^{\sigma}(e^{\lambda x} v(x))) \\ &= e^{-\lambda x} \frac{d^n}{dx^n} I^{n-\gamma+\sigma}(e^{\lambda x} v(x))) \\ &= e^{-\lambda x} {}_0^R\boldsymbol{\mathscr{D}_x}{}^{\gamma-\sigma}(e^{\lambda x} v(x))) \\ &= {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda} v(x), \end{aligned} \tag{11}$$

and

$$\begin{aligned} {}_0^R\mathbb{D}_x{}^{\sigma,\lambda} u(x)) &= e^{-\lambda x} {}_0^R\boldsymbol{\mathscr{D}_x}{}^{\sigma}(e^{\lambda x} u(x)) \\ &= e^{-\lambda x} {}_0^R\boldsymbol{\mathscr{D}_x}{}^{\sigma}(I^{\sigma}(e^{\lambda x} v(x))) \\ &= v(x). \end{aligned} \tag{12}$$

Thus, (11) and (12) yield that

$$-{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda} v(x) = f(x, I^{\sigma}(e^{\lambda x} v(x)), v(x)) - g(x, v(x)), \tag{13}$$

and

$$v(0) = {}_0^R\mathbb{D}_x{}^{\sigma,\lambda} u(0) = 0, \quad v(1) = {}_0^R\mathbb{D}_x{}^{\sigma,\lambda} u(1) = 0. \tag{14}$$

Thus, (13) and (14) indicate that Equation (3) is transformed into (10).

Contrarily, if $v \in C([0,1],[0,+\infty))$ is a positive solution of Equation (10), from Lemma 1 and (1), one has

$$\begin{aligned} -{}_0^R\mathbb{D}_x{}^{\gamma,\lambda} u(x)) - &= e^{-\lambda x} {}_0^R\boldsymbol{\mathscr{D}_x}{}^{\gamma}(e^{\lambda x} u(x)) \\ &= -e^{-\lambda x} \frac{d^n}{dx^n} I^{n-\gamma}(e^{\lambda x} u(x)) \\ &= -e^{-\lambda x} \frac{d^n}{dx^n} I^{n-\gamma} I^{\sigma}(e^{\lambda x} v(x)) \\ &= e^{-\lambda x} \frac{d^n}{dx^n} I^{n-\gamma+\sigma}(e^{\lambda x} v(x)) \\ &= -e^{-\lambda x} {}_0^R\boldsymbol{\mathscr{D}_x}{}^{\gamma-\sigma}(e^{\lambda x} v(x)) \\ &= -{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda} v(x) \\ &= f(x, I^{\sigma}(e^{\lambda x} v(x)), v(x)) - g(x, v(x)) \\ &= f(x, e^{\lambda x} u(x), {}_0^R\mathbb{D}_x{}^{\sigma,\lambda} u(x)) - g\left(x, {}_0^R\mathbb{D}_x{}^{\sigma,\lambda} u(x)\right), \quad 0 < x < 1. \end{aligned}$$

Since

$$ {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(x) = e^{-\lambda x}{}_0^R\boldsymbol{\mathscr{D}_x}{}^{\sigma}(e^{\lambda x}u(x)) = e^{-\lambda x}{}_0^R\boldsymbol{\mathscr{D}_x}{}^{\sigma}I^{\sigma}(e^{\lambda x}v(x)) = v(x), $$

one gets

$$ {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(0) = 0, \ {}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(1) = 0. $$

Noting that $v \in C([0,1],[0,+\infty))$ and the monotonicity of I^{σ}, one obtains

$$ I^{\sigma}(e^{\lambda x}v(x)) \in C([0,1],[0,+\infty)). $$

Hence, $u(x) = e^{-\lambda x}I^{\sigma}(e^{\lambda x}v(x))$ is a positive solution of Equation (3). □

Now, define a positive piecewise function of $\Lambda \in C[0,1]$ by

$$ [\Lambda(x)]^* = \begin{cases} \Lambda(x), & \Lambda(x) \geq 0, \\ 0, & \Lambda(x) < 0, \end{cases} $$

and make a translation transformation

$$ v(x) = z(x) - \ell(x). $$

We next focus on the modified problem of Equation (10)

$$ \begin{cases} -{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}z(x) = f(x, I^{\sigma}(e^{\lambda x}[z(x)-\ell(x)]^*), [z(x)-\ell(x)]^*) - g(x,[z(x)-\ell(x)]^*) + \hbar(x), \\ z(0) = 0, \ z(1) = 0. \end{cases} \tag{15} $$

Lemma 6. *Let z be a solution for the modified problem (15) satisfying $z(x) \geq \ell(x)$, $x \in [0,1]$. Then*

(i) *$z - \ell$ is a positive solution of Equation (10);*

(ii) *$u(x) = e^{-\lambda x}I^{\sigma}(e^{\lambda x}(z(x)-\ell(x)))$ is a positive solution of the Equation (3).*

Proof. Firstly, let z be a positive solution for the modified problem (15) with $z(x) \geq \ell(x)$, $x \in [0,1]$, then in view of the definition of $[\Lambda(x)]^*$ and (15), one has

$$ \begin{cases} -{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}z(x) = f(x, I^{\sigma}(e^{\lambda x}(z(x)-\ell(x))), z(x)-\ell(x)) - g(x, z(x)-\ell(x)) + \hbar(x), \\ z(0) = 0, \ z(1) = 0. \end{cases} \tag{16} $$

Letting $v = z - \ell$, and noting that

$$ \ell(0) = 0, \ \ell(1) = 0, \ z(0) = 0, \ z(1) = 0, $$

we have

$$ v(0) = 0, \ v(1) = 0, $$

and also have

$$ {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}v(x) = {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}z(x) - {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}\ell(x) = {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}z(x) + \hbar(x), $$

which implies that

$$ {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}z(x) = {}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}v(x) - \hbar(x). \tag{17} $$

Substituting (17) into (16), one has

$$ \begin{cases} -{}_0^R\mathbb{D}_x{}^{\gamma-\sigma,\lambda}v(x) = f(x, I^{\sigma}(e^{\lambda x}v(x)), v(x)) - g(x, v(x)), \\ v(0) = 0, \ v(1) = 0, \end{cases} $$

which indicate that $v = z - \ell$ solves Equation (10). As $z(x) \geq \ell(x)$, $v = z - \ell$ is positive. By Lemma 5, we know that $u(x) = e^{-\lambda x} I^\sigma(e^{\lambda x}(z(x) - \ell(x)))$ is a positive solution of the Equation (3). □

Now, define an operator T in E

$$(Tz)(x) = \int_0^1 G(x,y)\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy. \tag{18}$$

By Lemma 6, we only need to seek for the fixed points z of the operator T which satisfies $z(x) \geq \ell(x), x \in [0,1]$.

Lemma 7. *Assume* (**G1**) *and* (**G2**) *are satisfied. Then the operator $T : P \to P$ is completely continuous.*

Proof. For any fixed $z \in P$, one has $||z|| \leq M$ for some constant $M > 0$, and thus

$$\begin{aligned} &0 \leq [z(y)-\ell(y)]^* \leq z(y) \leq ||u|| \leq M, \\ &0 \leq I^\sigma(e^{\lambda y}[z(y)-\ell(y)]^*) = \int_0^x \frac{(x-y)^{\sigma-1}e^{\lambda y}[z(y)-\ell(y)]^*}{\Gamma(\sigma)}dy \leq \frac{Me^\lambda}{\Gamma(\sigma)}. \end{aligned} \tag{19}$$

It follows from (**G1**) and (**G2**) that T is continuous on $[0,1]$. Thus, in view of (19) and the continuity of f in $[0,1] \times [0, \frac{Me^\lambda}{\Gamma(\sigma)}] \times [0, M]$, we have

$$\begin{aligned} ||Tz|| &= \max_{x\in[0,1]} \int_0^1 G(x,y)\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy \\ &\leq \int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy \\ &\leq \int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)} f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) + 2\hbar(y)dy \\ &\leq \frac{e^\lambda}{\Gamma(\gamma-\sigma)}\left(\aleph + 2\int_0^1 \hbar(y)dy\right) < +\infty, \end{aligned} \tag{20}$$

where

$$\aleph = \max_{(x,u,v)\in[0,1]\times[0,\frac{Me^\lambda}{\Gamma(\sigma)}]\times[0,M]} f(x,u,v).$$

Thus, $T : P \to E$ is well defined.

Furthermore, it follows from (8) and (20) that

$$\begin{aligned} (Tz)(x) \geq &\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy \\ &\times x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \geq ||Tz||x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}, \end{aligned}$$

which implies that $T(P) \subset P$.

On the other hand, by using the standard arguments and combining Ascoli–Arzela theorem, $T : P \to P$ is continuous. Consequently, $T(P) \subset P$ is completely continuous. □

Lemma 8 ([58])**.** *Suppose P is a cone of a real Banach space E, the bounded open subsets Ω_1, Ω_2 of E satisfy $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|Tz\| \leq \|z\|, z \in P \cap \partial\Omega_1$ *and* $\|Tz\| \geq \|z\|, z \in P \cap \partial\Omega_2$, *or*

(2) $\|Tz\| \geq \|z\|, z \in P \cap \partial\Omega_1$ *and* $\|Tz\| \leq \|z\|, z \in P \cap \partial\Omega_2$.

Then, T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

In this section, we firstly define a constant

$$\kappa = \left(\frac{1}{\Gamma(\gamma-\sigma)} + \mu\right)e^{\lambda} \tag{21}$$

and then give a new result on the existence of positive solutions for Equation (3) under the case where the main nonlinearity f is superlinear. In addition, in order to ensure the existence of positive solutions for the sub-diffusion model with changing-sign perturbation, we shall also introduce a local control condition **(G5)** to serve our purpose.

(G5) Let μ and κ be defined by **(G1)** and (21), respectively. There exists a constant $r > \max\{2\kappa, 2\mu\}$ such that for any $(x,u,v) \in [0,1] \times [0,r] \times [0, \frac{re^{\lambda}}{\Gamma(\sigma)}]$,

$$f(x,u,v) \leq \frac{r-2\kappa}{\kappa}. \tag{22}$$

Theorem 1. *Suppose that* **(G1)**, **(G2)**, **(G3)** *and* **(G5)** *are satisfied. Then the singular sub-diffusion model involving the changing-sign perturbation (3) has at least one positive solution $u(x)$. Moreover, there exists a constant $K > 0$ such that*

$$u(x) \geq Ke^{-\lambda x}x^{\gamma-1}\left[1 - \frac{\gamma-\sigma}{\gamma}x\right], x \in [0,1].$$

Proof. Firstly, based on Lemma 6, it is sufficient to prove that there exists $z(x) \in C[0,1]$ with $z(x) \geq \ell(x)$ which solves the integro-differential tempered fractional Equation (15). To do this, we only need to prove that T has one fixed point $z(x)$ in P with $z(x) \geq \ell(x), x \in [0,1]$.

In fact, in view of Lemma 7, $T : P \to P$ is completely continuous. Now suppose $\Omega_1 = \{z \in E : ||z|| < r\}$, and $\partial\Omega_1 = \{z \in E : ||z|| = r\}$. For any $z \in P \cap \partial\Omega_1$, by the definition of $[\Lambda(x)]^*$, one has

$$\begin{aligned} &[z(y) - \ell(y)]^* \leq z(y) - \ell(y) \leq ||z|| = r, \\ &I^{\sigma}(e^{\lambda x}[z(y)-\ell(y)]^*) = \frac{1}{\Gamma(\sigma)}\int_0^x (x-y)^{\sigma-1}e^{\lambda y}[z(y)-\ell(y)]^* dy \leq \frac{re^{\lambda}}{\Gamma(\sigma)}. \end{aligned} \tag{23}$$

Consequently, by using **(G1)**, **(G5)**, (23), (8) and (22), we have

$$\begin{aligned} ||Tz|| &= \max_{x\in[0,1]} \int_0^1 G(x,y)\Big(f(y, I^{\sigma}(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy \\ &\leq \int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\Big(f(y, I^{\sigma}(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) + 2\hbar(y)\Big)dy \\ &\leq \int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\left(\frac{r-2\kappa}{\kappa} + 2\hbar(y)\right)dy \\ &\leq \int_0^1 \frac{re^{\lambda}}{\kappa\Gamma(\gamma-\sigma)}(1+\hbar(y))dy \\ &= \frac{re^{\lambda}}{\kappa}\left(\frac{1}{\Gamma(\gamma-\sigma)} + \mu\right) \\ &= r = ||z||. \end{aligned}$$

So, for any $z \in P \cap \partial\Omega_1$, we have $||Tz|| \leq ||z||$.

Next, take

$$\rho = 2\Gamma(\gamma-\sigma)\left\{a^{\gamma-\sigma-1}e^{\lambda(a-b)}(1-b)^{\gamma-\sigma}a\right\}^{-1}, \tag{24}$$

then for any $x \in [a,b]$, by **(G3)**, there exists a constant $N > r$ such that

$$f(x,u,v) > \rho(u+v), \tag{25}$$

for any $u+v > (\frac{a^\sigma}{\Gamma(\sigma+1)}+1)N$. Choose

$$R > \frac{2Ne^{\lambda b}}{a^{\gamma-\sigma-1}(1-b)} + r.$$

Noticing that $[a,b] \subset (0,1)$, one has

$$R > \frac{2Ne^{\lambda b}}{a^{\gamma-\sigma-1}(1-b)} + r > \frac{2Ne^{\lambda b}}{a^{\gamma-\sigma-1}(1-b)} > N > r > 2\mu.$$

Now, assume that $\Omega_2 = \{z \in E \mid ||z|| < R\}$ and $\partial\Omega_2 = \{z \in E \mid ||z|| = R\}$. In the following, we prove $||Tz|| \geq ||z||$, for $z \in P \cap \partial\Omega_2$.

In fact, for any $z \in P \cap \partial\Omega_2$, $x \in [a,b]$, one has

$$\begin{aligned} z(x) - \ell(x) &\geq z(x) - \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \geq z(x) - \frac{\mu z(x)}{||z||} = (1-\frac{\mu}{R})z(x) \\ &\geq \frac{1}{2}z(x) \geq \frac{1}{2}Rx^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \geq \frac{1}{2}Ra^{\gamma-\sigma-1}(1-b)e^{-\lambda b} \geq N > 0. \end{aligned} \tag{26}$$

$$\begin{aligned} I^\sigma(e^{\lambda x}(z(x)-\ell(x))) &= \frac{1}{\Gamma(\sigma)}\int_0^x (x-y)^{\sigma-1}e^{\lambda y}(z(y)-\ell(y))dy \\ &\geq \frac{N}{\Gamma(\sigma)}\int_0^x (x-y)^{\sigma-1}dy = \frac{N}{\Gamma(\sigma+1)}x^\sigma \geq \frac{N}{\Gamma(\sigma+1)}a^\sigma. \end{aligned} \tag{27}$$

Thus, for any $x \in [a,b]$, by (26) and (27), one has

$$I^\sigma(e^{\lambda x}(z(x)-\ell(x))) + z(x) - \ell(x) \geq \left(\frac{a^\sigma}{\Gamma(\sigma+1)}+1\right)N. \tag{28}$$

Hence, by employing (25) and (28), we obtain

$$\begin{aligned} ||Tz|| &= \max_{t\in[0,1]_{\mathbb{T}}} |\,(Tz)(x)]\,| \\ &= \max_{x\in[0,1]}\int_0^1 G(x,y)\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy \\ &\geq \max_{x\in[0,1]}\int_a^b G(x,y)\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*)\Big)dy \\ &\geq \max_{x\in[0,1]}\int_a^b G(x,y)\rho\Big(I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*) + [z(y)-\ell(y)]^*\Big)dy \\ &\geq \max_{x\in[0,1]}\int_a^b G(x,y)\rho[z(y)-\ell(y)]^*dy \\ &\geq \max_{x\in[0,1]}\int_a^b G(x,y)\frac{\rho R}{2}\times a^{\gamma-\sigma-1}(1-b)e^{-\lambda b}dy \\ &\geq \frac{\rho R}{2}\times a^{\gamma-\sigma-1}(1-b)e^{-\lambda b}\max_{x\in[0,1]}\int_a^b \frac{x^{\gamma-\sigma-1}(1-x)e^{-\lambda x}(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}dy \\ &\geq \frac{\rho R}{2\Gamma(\gamma-\sigma)}\times a^{\gamma-\sigma-1}e^{\lambda(a-b)}(1-b)^{\gamma-\sigma}a = R \\ &= ||z||, \end{aligned}$$

that is, $||Tz|| \geq ||z||$, $z \in P \cap \partial\Omega_2$. Thus, according to Lemma 8, T has a fixed point $z \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$ satisfying

$$2\mu \leq r \leq ||z|| \leq R.$$

Finally, we verify $z(x) \geq \ell(x), x \in [0,1]$. In fact,

$$\begin{aligned} z(x) - \ell(x) &\geq z(x) - \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \geq z(x) - \frac{\mu z(x)}{||z||} \geq \frac{1}{2}z(x) \\ &\geq \frac{1}{2}||z||x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \geq \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} > 0, x \in (0,1). \end{aligned} \tag{29}$$

Consequently, by Lemma 6, the singular tempered fractional Equation (3) has at least one positive solution $u(x)$

$$u(x) = e^{-\lambda x} I^{\sigma}(e^{\lambda x}(z(x) - \ell(x))).$$

Moreover, the positive solution $u(x)$ possesses an asymptotic property

$$\begin{aligned} &u(x) = e^{-\lambda x} I^{\sigma}(e^{\lambda x}(z(x) - \ell(x))) \geq \frac{e^{-\lambda x}}{\Gamma(\sigma)} \int_0^x (x-y)^{\sigma-1} e^{\lambda y}(z(y) - \ell(y))dy \\ &\geq \frac{\mu e^{-\lambda x}}{\Gamma(\sigma)} \int_0^x (x-y)^{\sigma-1} y^{\gamma-\sigma-1}(1-y)dy \geq \frac{\mu e^{-\lambda x}}{\Gamma(\sigma)} \left[x^{\gamma-1} \frac{\Gamma(\sigma)\Gamma(\gamma-\sigma)}{\Gamma(\gamma)} - x^{\gamma} \frac{\Gamma(\sigma)\Gamma(\gamma-\sigma+1)}{\Gamma(\gamma+1)} \right] \\ &= \frac{\Gamma(\gamma-\sigma)\mu e^{-\lambda x} x^{\gamma-1}}{\Gamma(\gamma)} \left[1 - \frac{\gamma-\sigma}{\gamma} x \right] = K e^{-\lambda x} x^{\gamma-1} \left[1 - \frac{\gamma-\sigma}{\gamma} x \right], \end{aligned} \tag{30}$$

where

$$K = \frac{\Gamma(\gamma-\sigma)\mu}{\Gamma(\gamma)}.$$

□

For the sublinear case for nonlinearity f, the following existence result has been derived.

Theorem 2. *Suppose that* ***(G1)***, ***(G2)*** *and* ***(G4)*** *hold. Moreover* ***(G6)*** *holds, that is:*

There exist $[a,b] \subset (0,1)$ such that for any $(x,u,v) \in [a,b] \times [\mu a^{\gamma-\sigma-1}(1-b)e^{-\lambda b}, 2\mu] \times [\frac{\mu\Gamma(\gamma-\sigma)}{\Gamma(\gamma)}(1-b)a^{\gamma-1}, \frac{2\mu e^{\lambda}}{\Gamma(\sigma)}]$, we have

$$f(x,u,v) \geq \frac{2\mu\Gamma(\gamma-\sigma)}{a^{\gamma-\sigma}(1-b)^{\gamma-\sigma}(b-a)}.$$

Then, the singular sub-diffusion model with a changing-sign perturbation (3) has at least one positive solution $u(x)$. Moreover, there exists a constant $K > 0$ such that

$$u(x) \geq K e^{-\lambda x} x^{\gamma-1} \left[1 - \frac{\gamma-\sigma}{\gamma} x \right], x \in [0,1].$$

Proof. Firstly, Lemma 7 still implies that $T(P) \subset P$ is completely continuous.

Take $\Omega_3 = \{z \in K : ||z|| < 2\mu\}$ and $\partial\Omega_3 = \{z \in K : ||z|| = 2\mu\}$. Then, for any $z \in P \cap \partial\Omega_3$ and $t \in [0,1]$, we have

$$\begin{aligned} & 0 < \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \le \frac{1}{2}||z||x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \le \frac{1}{2}z(x) \\ & = z(x) - \frac{\mu z(x)}{||z||} \le z(x) - \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \\ & \le z(x) - \ell(x) \\ & \le z(x) \le ||z|| = 2\mu. \end{aligned} \tag{31}$$

and

$$\begin{aligned} & 0 < \frac{\mu\Gamma(\gamma-\sigma)}{\Gamma(\gamma)}(1-x)x^{\gamma-1} \le \int_0^x \frac{(x-y)^{\sigma-1}\mu y^{\gamma-\sigma-1}(1-y)}{\Gamma(\sigma)}dy \\ & \le \int_0^x \frac{(x-y)^{\sigma-1}e^{\lambda y}[z(y)-\ell(y)]^*}{\Gamma(\sigma)}dy \\ & = I^\sigma(e^{\lambda y}[z(y)-\ell(y)]^*) \le \frac{2\mu e^\lambda}{\Gamma(\sigma)}. \end{aligned} \tag{32}$$

Thus, by (31) and (32), for any $z \in P \cap \partial\Omega_3$ and $x \in [a,b]$, one has

$$\mu a^{\gamma-\sigma-1}(1-b)e^{-\lambda b} \le z(x) - \ell(x) \le 2\mu, \tag{33}$$

and

$$\frac{\mu\Gamma(\gamma-\sigma)}{\Gamma(\gamma)}(1-b)a^{\gamma-1} \le I^\sigma(e^{\lambda y}[z(y)-\ell(y)]^*) \le \frac{2\mu e^\lambda}{\Gamma(\sigma)}. \tag{34}$$

Consequently, for any $z \in P \cap \partial\Omega_3$, by (33), (34) and **(G6)**, one has

$$\begin{aligned} & ||Tz|| \ge \max_{x\in[0,1]_{\mathbb{T}}} |\,(Tz)(x)]\,| \\ & = \max_{x\in[0,1]} \int_0^1 G(x,y)\Big(f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*) + \hbar(y)\Big)dy \\ & \ge \int_a^b G(a,y)f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*)dy \\ & \ge \frac{2\mu\Gamma(\gamma-\sigma)}{a^{\gamma-\sigma}(1-b)^{\gamma-\sigma}(b-a)} \int_a^b G(a,y)dy \\ & \ge \frac{2\mu\Gamma(\gamma-\sigma)}{a^{\gamma-\sigma}(1-b)^{\gamma-\sigma}(b-a)} \int_a^b \frac{a^{\gamma-\sigma-1}(1-a)e^{-\lambda a}(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}dy \\ & \ge 2\mu = ||z||, \end{aligned}$$

i.e., $||Tz|| \ge ||z||$ for any $z \in P \cap \partial\Omega_3$.

Next, take $\varepsilon > 0$ sufficiently small with

$$\varepsilon\left(\frac{e^\lambda}{\Gamma(\sigma)} + 1\right)\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}dy < 1.$$

For the above ε and for any $x \in [0,1]$, by (**G4**), there exists $N > 2\mu > 0$ such that

$$f(x,u,v) \le \varepsilon|u+v|, \text{ if } |u+v| > N. \tag{35}$$

Thus, (35) implies that if $\left|I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*)\right| + |[z(y)-\ell(y)]^*| > N$ holds, then we have

$$\begin{aligned} & f(y, I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*), [z(y)-\ell(y)]^*)dy \\ & \qquad \le \left|I^\sigma(e^{\lambda x}[z(y)-\ell(y)]^*) + [z(y)-\ell(y)]^*\right|\varepsilon. \end{aligned}$$

It follows from (31) and (32) that

$$f(y,I^{\sigma}(e^{\lambda x}[z(y)-\ell(y)]^*),[z(y)-\ell(y)]^*)dy \le \left(\frac{e^{\lambda}}{\Gamma(\sigma)}+1\right)||z||\varepsilon.$$

Take

$$R^* = \frac{\kappa\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}(1+\hbar(y))dy + 2\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\hbar(y)dy}{1-\varepsilon\left(\frac{e^{\lambda}}{\Gamma(\sigma)}+1\right)\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}dy} + N,$$

where

$$\kappa = \max_{\substack{x\in[0,1]\\ |u+v|\le N}} f(x,u,v)+2.$$

Then, $R^* > N > 2\mu$.

Assume $\Omega_4 = \{z \in P : ||z|| < R^*\}$ and $\partial\Omega_4 = \{z \in P : ||z|| = R^*\}$. Then, for any $z \in P \cap \partial\Omega_4$, one has

$$\begin{aligned}
||Tu|| &= \max_{x\in[0,1]} \int_0^1 G(x,y)\Big(f(y,I^{\sigma}(e^{\lambda x}[z(y)-\ell(y)]^*),[z(y)-\ell(y)]^*) - g(y,[z(y)-\ell(y)]^*)+\hbar(y)\Big)dy \\
&\le \int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\Big(f(y,I^{\sigma}(e^{\lambda x}[z(y)-\ell(y)]^*),[z(y)-\ell(y)]^*)+2\hbar(y)\Big)dy \\
&\le \left(\max_{\substack{x\in[0,1]\\ |u+v|\le N}} f(x,u,v)+2\right)\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}(1+\hbar(y))dy \\
&\quad + \int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}[\varepsilon\left(\frac{e^{\lambda}}{\Gamma(\sigma)}+1\right)||z||+2\hbar(y)]dy \\
&\le \kappa\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}(1+\hbar(y))dy + 2\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}\hbar(y)dy \\
&\quad + \varepsilon\left(\frac{e^{\lambda}}{\Gamma(\sigma)}+1\right)\int_0^1 \frac{(1-y)^{\gamma-\sigma-1}ye^{\lambda y}}{\Gamma(\gamma-\sigma)}dyR^* < R^* = ||z||,
\end{aligned}$$

that is

$$||Tz|| \le ||z||, z \in P\cap\partial\Omega_4.$$

Therefore, Lemma 8 ensures that T has one fixed point $z \in P\cap(\overline{\Omega}_4 \setminus \Omega_3)$ satisfying

$$2\mu \le ||z|| \le R^*.$$

We assert that $z(x) \ge \ell(x),\ \ x \in [0,1]$. In fact, by (9) and $z \in P$, one gets

$$\begin{aligned}
z(x)-\ell(x) &\ge z(x) - \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} \ge z(x) - \frac{\mu z(x)}{||z||} \ge \frac{1}{2}z(x) \\
&\ge \mu x^{\gamma-\sigma-1}(1-x)e^{-\lambda x} > 0, x \in (0,1).
\end{aligned} \tag{36}$$

Thus, Lemma 6 guarantees that $u(x) = e^{-\lambda x}I^{\sigma}(e^{\lambda x}(z(x)-\ell(x)))$ is a positive solution of the singular sub-diffusion model with a changing-sign perturbation (3). Moreover, similar to (30), there is a constant $K > 0$ such that

$$u(x) \ge Ke^{-\lambda x}x^{\gamma-1}\left[1-\frac{\gamma-\sigma}{\gamma}x\right].$$

□

4. Examples

Let
$$\gamma=\frac{3}{2},\ \sigma=\frac{1}{2},\ \lambda=2,$$
we give four examples for the cases in which the main nonlinear term f is superlinear or sublinear and the negative perturbation g may tend to negative infinity.

Example 1. *Consider the singular superlinear sub-diffusion model with a changing-sign perturbation*
$$\begin{cases} -{}_0^R\mathbb{D}_x{}^{\frac{3}{2},2}u(x)=2.1641\times10^{-7}\left(e^{2x}u(x)+{}_0^R\mathbb{D}_x{}^{\frac{1}{2},2}u(x)\right)^2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}-\dfrac{1}{(\frac{1}{2}-x)^{\frac{2}{3}}\left(e^{\left({}_0^R\mathbb{D}_x{}^{\frac{1}{2},2}u(x)\right)^2}+1\right)}, \\ {}_0^R\mathbb{D}_x{}^{\frac{1}{2},2}u(0)=0,\ {}_0^R\mathbb{D}_x{}^{\frac{1}{2},2}u(1)=0. \end{cases} \tag{37}$$

Conclusion. The singular superlinear sub-diffusion model with a changing-sign perturbation (37) has at least one positive solution $u(x)$ with the following asymptotic property
$$u(x)\geq 5.3736e^{-2x}x^{\frac{1}{2}}\left(1-\frac{2}{3}x\right),x\in[0,1].$$

Proof. Denote
$$f(x,u,v)=2.1641\times10^{-7}(u+v)^2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}},\ \ g(x,v)=\frac{1}{(\frac{1}{2}-x)^{\frac{2}{3}}(e^{v^2}+1)}.$$

It is clear that $f:[0,1]\times[0,+\infty)\times(-\infty,+\infty)\to[0,+\infty)$ is continuous, and $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies the Crathèodory condition with
$$\hbar(x)=\frac{1}{(\frac{1}{2}-x)^{\frac{2}{3}}}$$
and
$$\mu=\frac{1}{\Gamma(\gamma-\sigma)}\int_0^1\hbar(y)dy=\int_0^1\frac{1}{(\frac{1}{2}-y)^{\frac{2}{3}}}dy=6\times(\frac{1}{2})^{\frac{1}{3}}=4.7622.$$

Thus, **(G1)–(G3)** are satisfied.

Now, we verify the condition **(G5)**. Firstly, we have
$$\kappa=\left(\frac{1}{\Gamma(\gamma-\sigma)}+\mu\right)e^{\lambda}=(1+\mu)e^2=43.5772.$$

Take $r=90>\max\{85.1544,9.5244\}=85.1544$. Then, for any $(x,u,v)\in[0,1]\times[0,r]\times[0,\frac{re^{\lambda}}{\Gamma(\sigma)}]=[0,1]\times[0,90]\times[0,375.1941]$, we have
$$\begin{aligned} f(x,u,v)&=2.1641\times10^{-7}(u+v)^2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\leq 2.1641\times10^{-7}\left(90+\frac{90e^2}{\Gamma(\frac{1}{2})}\right)^2=0.01 \\ &\leq\frac{r-2\kappa}{\kappa}=\frac{90-85.1544}{42.5772}=0.1138, \end{aligned}$$
i.e., **(G5)** is satisfied. Next, as
$$K=\frac{\Gamma(\gamma-\sigma)\mu}{\Gamma(\gamma)}=\frac{4.7622}{\Gamma(\frac{3}{2})}=5.3736,$$

one has

$$u(x) \geq 5.3736 e^{-2x} x^{\frac{1}{2}} \left(1 - \frac{2}{3}x\right), x \in [0,1].$$

By Theorem 1, the singular superlinear sub-diffusion model involving the changing-sign perturbation (37) has at least one positive solution $u(x)$ satisfying

$$u(x) \geq 5.3736 e^{-2x} x^{\frac{1}{2}} \left(1 - \frac{2}{3}x\right), x \in [0,1].$$

□

Example 2. *Consider the sublinear sub-diffusion model with a changing-sign perturbation*

$$\begin{cases} -{}_0^R\mathbb{D}_x^{\frac{3}{2},2} u(x) = 3000\left(e^{2x}u(x) + {}_0^R\mathbb{D}_x^{\frac{1}{2},2}u(x)\right)^{\frac{1}{2}} x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} - \dfrac{1}{(\frac{1}{2}-x)^{\frac{2}{3}}\left(e^{\left({}_0^R\mathbb{D}_t^{\frac{1}{2},2}u(x)\right)^2} + 1\right)}, \\ {}_0^R\mathbb{D}_x^{\frac{1}{2},2}u(0) = 0, \ {}_0^R\mathbb{D}_x^{\frac{1}{2},2}u(1) = 0. \end{cases} \tag{38}$$

Conclusion. the sublinear sub-diffusion model involving the changing-sign perturbation (38) has at least one positive solution $u(x)$ satisfying

$$u(x) \geq 5.3736 e^{-2x} x^{\frac{1}{2}} \left(1 - \frac{2}{3}x\right), x \in [0,1].$$

Proof. Denote

$$f(x,u,v) = 3000(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}, \ g(x,v) = \frac{1}{(\frac{1}{2}-x)^{\frac{2}{3}}(e^{v^2}+1)}.$$

It is clear that $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous, and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Crathèodory condition with

$$\hbar(x) = \frac{1}{(\frac{1}{2}-x)^{\frac{2}{3}}}$$

and

$$\mu = \frac{1}{\Gamma(\gamma-\sigma)} \int_0^1 \hbar(y)dy = \int_0^1 \frac{1}{(\frac{1}{2}-y)^{\frac{2}{3}}} dy = 6 \times \left(\frac{1}{2}\right)^{\frac{1}{3}} = 4.7622.$$

Thus, **(G1)**, **(G2)** and **(G4)** are satisfied.

Now, we verify the condition **(G6)**. Take a compact interval $[\frac{1}{4}, \frac{3}{4}] \subset (0,1)$, then for any $(x,u,v) \in [a,b] \times [\mu a^{\gamma-\sigma-1}(1-b)e^{-\lambda b}, 2\mu] \times [\frac{\mu\Gamma(\gamma-\sigma)}{\Gamma(\gamma)}(1-b)a^{\gamma-1}, \frac{2\mu e^{\lambda}}{\Gamma(\sigma)}] = [\frac{1}{4}, \frac{3}{4}] \times [4.7622 \times (\frac{1}{4})e^{-\frac{3}{2}}, 2 \times 4.7622] \times [\frac{4.7622}{\Gamma(\frac{3}{2})}(\frac{1}{4})^{\frac{3}{2}}, \frac{2\times 4.7622 e^2}{\Gamma(\frac{1}{2})}] = [\frac{1}{4}, \frac{3}{4}] \times [0.2657, 9.5244] \times [0.6717, 39.7055]$, we have

$$f(x,u,v) \geq 3000 \times (0.2657 + 0.6717)^{\frac{1}{2}} \times \frac{1}{4} = 703.05 \geq \frac{2\mu\Gamma(\gamma-\sigma)}{a^{\gamma-\sigma}(1-b)^{\gamma-\sigma}(b-a)} = \frac{2 \times 4.7622}{\frac{1}{64}} = 609.5616.$$

Thus, **(G6)** holds.

In addition, we still have

$$K = \frac{\Gamma(\gamma-\sigma)\mu}{\Gamma(\gamma)} = \frac{4.7622}{\Gamma(\frac{3}{2})} = 5.3736.$$

Consequently,

$$u(x) \geq 5.3736 e^{-2x} x^{\frac{1}{2}} \left(1 - \frac{2}{3}x\right), x \in [0,1].$$

Then, by Theorem 2, the sublinear sub-diffusion model involving the changing-sign perturbation (38) has at least one positive solution $u(x)$ satisfying

$$u(x) \geq 5.3736 e^{-2x} x^{\frac{1}{2}} \left(1 - \frac{2}{3}x\right), x \in [0,1].$$

□

Example 3. *Consider the singular superlinear sub-diffusion model with a changing-sign perturbation*

$$\begin{cases} -{}_0^R\mathbb{D}_x{}^{\frac{3}{2},2} u(x) = f\left(x, e^{\lambda x} u(x), {}_0^R\mathbb{D}_x{}^{\sigma,\lambda} u(x)\right) - \dfrac{1}{3x^{\frac{1}{2}}}, \\ {}_0^R\mathbb{D}_x{}^{\frac{1}{2},2} u(0) = 0, \ {}_0^R\mathbb{D}_x{}^{\frac{1}{2},2} u(1) = 0, \end{cases} \tag{39}$$

where

$$f(x,u,v) = \begin{cases} \dfrac{(u+v)^{\frac{1}{2}}}{1527.89}, & (x,u,v) \in [0,1] \times [0,25] \times [0,208.4462] \\ \dfrac{(u+v)^2}{5.4497 \times 10^6}, & x \in [0,1], u > 25, v > 208.4462. \end{cases} \tag{40}$$

Conclusion. The singular superlinear sub-diffusion model with a changing-sign perturbation (39) has at least one positive solution $u(x)$ with the following asymptotic property

$$u(x) \geq 0.7523 e^{-2x} x^{\frac{1}{2}} \left(1 - \frac{2}{3}x\right), x \in [0,1].$$

Proof. Let $f(x,u,v)$ be as defined in (40) and

$$g(x,v) = \frac{1}{3x^{\frac{1}{2}}}.$$

Then, $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous, and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Crathèodory condition with

$$\hbar(x) = \frac{1}{3x^{\frac{1}{2}}}$$

and

$$\mu = \frac{1}{\Gamma(\gamma-\sigma)} \int_0^1 \hbar(y)dy = \int_0^1 \frac{1}{3y^{\frac{1}{2}}} dy = \frac{2}{3}.$$

Thus, **(G1)**, **(G2)** and **(G3)** are satisfied.

Now, we verify the condition **(G5)**. Firstly, we have

$$\kappa = \left(\frac{1}{\Gamma(\gamma-\sigma)} + \mu\right) e^{\lambda} = (1+\mu)e^2 = 12.3153.$$

Choose $r = 25 > \max\{24.6306, \frac{4}{3}\} = 24.6306$. Then, for any $(x,u,v) \in [0,1] \times [0,r] \times [0, \frac{re^{\lambda}}{\Gamma(\sigma)}] = [0,1] \times [0,25] \times [0,208.4462]$, we have

$$f(x,u,v) = \frac{(u+v)^{\frac{1}{2}}}{1527.89} \leq 0.01 \leq \frac{r-2\kappa}{\kappa} = \frac{25-24.6306}{12.3153} = 0.03,$$

which implies that **(G5)** holds.

Next, as

$$K = \frac{\Gamma(\gamma-\sigma)\mu}{\Gamma(\gamma)} = \frac{\frac{2}{3}}{\Gamma(\frac{3}{2})} = 0.7523,$$

one has

$$u(x) \geq 0.7523e^{-2x}x^{\frac{1}{2}}\left(1-\frac{2}{3}x\right), x \in [0,1].$$

By Theorem 1, the singular superlinear sub-diffusion model involving the changing-sign perturbation (39) has at least one positive solution $u(x)$ satisfying

$$u(x) \geq 0.7523e^{-2x}x^{\frac{1}{2}}\left(1-\frac{2}{3}x\right), x \in [0,1].$$

□

Example 4. *Consider the singular sub-linear sub-diffusion model with a changing-sign perturbation*

$$\begin{cases} -{}_0^R\mathbb{D}_x{}^{\frac{3}{2},2}u(x) = 3000e^{-\frac{1}{e^{\lambda x}u(x)+{}_0^R\mathbb{D}_x{}^{\sigma,\lambda}u(x)}} - \dfrac{1}{(1-x)^{\frac{2}{3}}}, \\ {}_0^R\mathbb{D}_x{}^{\frac{1}{2},2}u(0) = 0, \ \ {}_0^R\mathbb{D}_x{}^{\frac{1}{2},2}u(1) = 0. \end{cases} \tag{41}$$

Conclusion. The singular sublinear sub-diffusion model with a changing-sign perturbation (41) has at least one positive solution $u(x)$ with the following asymptotic property

$$u(x) \geq 3.3852e^{-2x}x^{\frac{1}{2}}\left(1-\frac{2}{3}x\right), x \in [0,1].$$

Proof. Let

$$f(x,u,v) = 3000e^{-\frac{1}{u+v}}, \ \ g(x,v) = \frac{1}{(1-x)^{\frac{2}{3}}}.$$

Then, $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous, and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Crathèodory condition with

$$\hbar(x) = \frac{1}{(1-x)^{\frac{2}{3}}}$$

and

$$\mu = \frac{1}{\Gamma(\gamma-\sigma)}\int_0^1 \hbar(y)dy = \int_0^1 \frac{1}{(1-y)^{\frac{2}{3}}}dy = 3.$$

Thus, **(G1)**, **(G2)** and **(G4)** are satisfied.

Now, we verify the condition **(G6)**. Take a compact interval $[\frac{1}{4},\frac{3}{4}] \subset (0,1)$, then for any $(x,u,v) \in [a,b] \times [\mu a^{\gamma-\sigma-1}(1-b)e^{-\lambda b}, 2\mu] \times [\frac{\mu\Gamma(\gamma-\sigma)}{\Gamma(\gamma)}(1-b)a^{\gamma-1}, \frac{2\mu e^{\lambda}}{\Gamma(\sigma)}] = [\frac{1}{4},\frac{3}{4}] \times [\frac{3}{4}e^{-\frac{3}{2}}, 6] \times [\frac{3}{\Gamma(\frac{3}{2})}(\frac{1}{4})^{\frac{3}{2}}, \frac{6e^2}{\Gamma(\frac{1}{2})}] = [\frac{1}{4},\frac{3}{4}] \times [0.1673, 6] \times [0.4232, 25.0121]$, we have

$$f(x,u,v) = 3000e^{-\frac{1}{u+v}} \geq 3000 \times 0.1839 = 551.7 \geq \frac{2\mu\Gamma(\gamma-\sigma)}{a^{\gamma-\sigma}(1-b)^{\gamma-\sigma}(b-a)} = \frac{6}{\frac{1}{64}} = 384.$$

Thus, **(G6)** holds.

In addition, we still have

$$K = \frac{\Gamma(\gamma-\sigma)\mu}{\Gamma(\gamma)} = \frac{3}{\Gamma(\frac{3}{2})} = 3.3852.$$

Consequently,

$$u(x) \geq 3.3852e^{-2x}x^{\frac{1}{2}}\left(1 - \frac{2}{3}x\right), x \in [0,1].$$

Then, by Theorem 2, the sublinear sub-diffusion model involving the changing-sign perturbation (41) has at least one positive solution $u(x)$ satisfying

$$u(x) \geq 3.3852e^{-2x}x^{\frac{1}{2}}\left(1 - \frac{2}{3}x\right), x \in [0,1].$$

□

5. Conclusions

In anomalous diffusion, the mean square variance sometimes grows faster to create a super-diffusion and sometimes spreads slower to form sub-diffusion than in the Gaussian process. The anomalous sub-diffusion in Brownian motion can be modeled by tempered fractional diffusion equation. This paper focused on a singular tempered sub-diffusion fractional equation involving a changing-sign perturbation and lower-order sub-diffusion term of unknown functions. By employing multiple transformations, the changing-sign singular perturbation problem was converted to a positive problem, and then some sufficient conditions for the existence of positive solutions of the problem and the asymptotic properties of solution were derived by employing the fixed-point theorem. In particular, the singular perturbation term only satisfies the Carathéodory condition, which makes the disturbance influence to be significant and even achieve the negative infinity near some time singular points. Thus, our main results can be applied to handle some changing-sign anomalous sub-diffusion models. However the problem for the case where changing-sign perturbation does not satisfy the Carathéodory condition still remains an open problem for further research. In future work, we will continue to focus on the existence of solutions for other types of changing-sign problems.

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