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Quasi-Contraction Maps in Subordinate Semimetric Spaces

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Abstract: Throughout this study, we discuss the subordinate Pompeiu–Hausdorff metric (SPHM) in subordinate semimetric spaces. Moreover, we present a well-behaved quasi-contraction (WBQC) to solve quasi-contraction (QC) problems in subordinate semimetric spaces under some local constraints. Furthermore, we provide examples to support our conclusion.

Keywords: subordinate semimetric space; iterative fixed point; strict fixed point; subordinate Pompeiu–Hausdorff metric; well-behaved quasi-contraction

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1. Introduction

Assume (Γ, Ψ) is a metric space. A function $T : \Gamma \to \Gamma$ is said to be quasi-contraction (QC) if there is a constant $\alpha \in (0, 1)$ such that for each $\omega, \nu \in \Gamma$,

$$\Psi(T\omega, T\nu) \leq \alpha \max\{\Psi(\omega, \nu), \Psi(\omega, T\omega), \Psi(\nu, T\nu), \Psi(\omega, T\nu), \Psi(\nu, T\omega)\}.$$

Ćirić [1] was the first to introduce and study this concept as one of the most basic contractive type functions. The recognized Ćirić's theorem indicates that a QC T has a unique fixed point on a complete metric space Γ.

Let (Γ, Ψ) be a complete distance space and let $T : \Gamma \to P_{cb}(\Gamma)$ be a set-valued quasi-contraction (SVQC) for $\frac{1}{2} < \alpha < 1$. Is there a fixed point for T?

Following the method of Pourrazi, Khojasteh, Javahernia, and Khandani [2], we will try to solve this problem in a subordinate semimetric space setting.

In 2018, Villa-Morlales [3] presented subordinate semi-metric spaces that include a wide range of distance spaces, such as a JS-metric spaces, standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces [3,4]. Looking over the literature that includes a subordinate semimetric space, we can see that the Hausdorff metric, which is created by a subordinate semimetric space, still needs to be examined. We were prompted to propose the SVQC and solve the above problem in these structures.

1.1. Semimetric Spaces

Definition 1. *Let Γ be a nonempty set and $\Psi : \Gamma \times \Gamma \to [0, +\infty]$ be a nonnegative and symmetric function which vanishes exactly on the diagonal of $\Gamma \times \Gamma$. Thus (Γ, Ψ) is called a semimetric space.*

Definition 2. *Let (Γ, Ψ) be a semimetric space and let $\{\omega_n\}$ be a sequence in Γ and $\omega \in \Gamma$. Then,*

(i) $\{\omega_n\}$ is convergent to ω if $\lim_{n\to\infty} \Psi(\omega, \omega_n) = 0$.
(ii) $\{\omega_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} \Psi(\omega_n, \omega_m) = 0$.
(iii) The pair (Γ, Ψ) is complete if every Cauchy sequence in Γ is convergent.
(iv) For every $\epsilon > 0$, a ball is defined as $B(\omega_0, \epsilon) = \{\omega \in \Gamma \mid \Psi(\omega_0, \omega) < \epsilon\}$.
(v) A diameter of a set $\Delta \subset \Gamma$, is $\delta(\Delta) = \sup\{\Psi(\mu, \nu) \mid \mu, \nu \in \Delta\}$.

(vi) *A set $\Delta \subset \Gamma$ is bounded if $\delta(\Delta) < \infty$.*

With these notations, we have two ways to define a topology on Γ:

(1) The neighborhood topology: Let $\Delta \subseteq \Gamma$. An element $\nu \in \Delta$ is called an interior point of Δ if there exists $\epsilon > 0$, such that $B(\nu, \epsilon) \subseteq \Delta$. Let Δ° be the set of all interior points of Δ. A set $\Delta \subseteq \Gamma$ is open if $\Delta^\circ = \Delta$.

(2) The sequential topology: Let $\Delta \subseteq \Gamma$ be a nonempty set and $\omega \in \Gamma$ be a point. We say ω is a closure point of Δ if there exists a sequence $\{\omega_n\} \subset \Delta$ such that $\lim_{n\to\infty} \Psi(\omega_n, \omega) = 0$. Let $\overline{\Delta}$ be the set of closure points of Δ. Then a set Δ is closed if $\Delta = \overline{\Delta}$.

Note that in metric spaces, the complement of every sequentially closed set is topologically open. On the other hand, in semimetric spaces, this property may not hold in general. The two topologies may not be the same. The limit of a sequence is not necessarily unique; a convergent sequence is not necessarily a Cauchy sequence.

Definition 3. *Let (Γ, Ψ) be a semimetric space. We say that $\Phi : [0, \infty] \times [0, \infty] \to [0, \infty]$ is a triangle function for Ψ, if Φ is symmetric and monotone increasing in both of its arguments such that $\Phi(0, 0) = 0$ and for all $\omega, \nu, \mu \in \Gamma$, $\Psi(\omega, \nu) \leq \Phi(\Psi(\omega, \mu), \Psi(\mu, \nu))$.*

It turns out that every semimetric space has an optimal (with respect to the pointwise ordering) triangle function which is called the basic triangle function. A semimetric space is called normal if its basic triangle function is real-valued and is called regular if its basic triangle function is continuous at $(0, 0)$.

In [5,6], the authors proved the following Theorems:

Theorem 1. *A semimetric space (Γ, Ψ) is regular if and only $\lim_{\epsilon\to 0} \sup_{\nu\in\Gamma} B(\nu, \epsilon) = 0$. Furthermore, in a regular semimetric space, convergent sequences have a unique limit and possess the Cauchy property.*

Theorem 2. *Let (Γ, Ψ) be a semimetric space. Let $\Delta \subseteq \Gamma$ be a closed set and $\{\omega_n\}$ be a sequence in Δ such that $\lim_{n\to\infty} \Psi(\omega_n, \omega) = 0$ for some $\omega \in \Gamma$, then $\omega \in \Delta$. If the semimetric space is also regular, then, for every $\epsilon > 0$, there exists $r > 0$, such that, for all $\omega \in \Gamma$, the inclusion $B(\omega, r) \subseteq B(\omega, \epsilon)^\circ$. holds; consequently, the topology is Hausdorff.*

One of the main results of [7] characterizes regular semimetric spaces in term of uniform equivalence.

We say that the semimetrics Ψ and Υ on Γ are uniformly equivalent if $id : (\Gamma, \Psi) \to (\Gamma, \Upsilon)$ is uniformly bi-continuous. That is,

(i) for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\Psi(\omega, \nu) < \delta \Rightarrow \Upsilon(\omega, \nu) < \epsilon$; and
(ii) for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\Upsilon(\omega, \nu) < \delta \Rightarrow \Psi(\omega, \nu) < \epsilon$;

The fundamental result goes as follows.

Theorem 3. *A semimetric space is regular if and only if it is uniformly equivalent to a metric.*

Thus in regular semimetric spaces the neighborhood and sequential topologies coincide and Hausdorff property holds. We will consider regular semimetric spaces on our study.

1.2. Subordinate Semimetric Spaces

Definition 4 ([3])**.** *Ψ is said to be a subordinate semimetric on Γ if it meets the following conditions: for every $(\rho, \mu) \in \Gamma^2$,*

(D1) If $\Psi(\rho, \mu) = 0$ then $\rho = \mu$,
(D2) $\Psi(\rho, \mu) = \Psi(\mu, \rho)$.

(D3) There is a function $\xi : [0, \infty] \to [0, \infty]$ satisfies

(i) ξ *is non-decreasing;*

(ii) $\lim_{\rho \to 0} \xi(\rho) = 0$;

such that for every $(\rho, \mu) \in \Gamma^2$, with $\rho \neq \mu$, and for any infinite Cauchy sequence $\{\rho_n\}$ in Γ that converges to ρ we have

$$\Psi(\rho, \mu) \leq \xi(\limsup_{n \to \infty} \Gamma(\rho_n, \mu)).$$

The space (Γ, Ψ) is called subordinate to ξ or (Γ, Ψ) is a subordinate semimetric space.

In this paper, we introduce Well-Behaved Quasi-Contraction (WBQC) set-valued mapping (SVM) in subordinate semi-metric space and obtained at least one fixed point when $\alpha \in (0, 1)$. Finally, we provide significant examples. Inspired by the characterization of completeness of b-metric spaces in [8]. Villa-Morlales in [3] characterize when a subordinate semimetric space is complete.

2. Subordinate Pompeiu Hausdorff Metric Spaces

Assume (Γ, ϱ) is a metric space, and the set of all nonempty, bounded, and closed subsets of Γ denoted by $P_{cb}(\Gamma)$. Assume $T : \Gamma \to P_{cb}(\Gamma)$ is an SVM on Γ. An element $\rho \in \Gamma$ is said to be a fixed point of T if $\rho \in T\rho$. Consider $\mathcal{F}(T) = \{\rho \in \Gamma : \rho \in T\rho\}$. We say that a point $\rho \in \Gamma$ is a strict fixed point of T, if $T\rho = \{\rho\}$. The family of all strict fixed points of T are denoted by $\mathcal{SF}(T) = \{\rho \in \Gamma : T\rho = \{\rho\}\}$. Then, we have $\mathcal{SF}(T) \subseteq \mathcal{F}(T)$.

For further knowledge of the development of fixed point theory in the family of set-valued mappings (SVMs), we recommend [9]. Assume H is the Pompeiu–Hausdorff metric (PHM) [10] on $P_{cb}(\Gamma)$ produced by ϱ, as,

$$\mathcal{H}(A, B) = \max\{\sup_{\rho \in B} \varrho^*(\rho, A), \sup_{\rho \in A} \varrho^*(\rho, B)\}, A, B \in P_{cb}(\Gamma)$$

such that, $\varrho^*(\rho, A) = \inf\{\Psi(\rho, \mu) : \mu \in A\}$.

Definition 5 ([2])**.** *Let (Γ, Ψ) be a subordinate semimetric space. Let $\mathcal{H}_\Psi$ be defined by*

$$\mathcal{H}_\Psi(A, B) = \max\{\sup_{\rho \in B} \Psi^*(\rho, A), \sup_{\rho \in A} \Psi^*(\rho, B)\}, A, B \in P_{cb}(\Gamma)$$

in which, $\Psi^*(\rho, A) = \inf\{\Psi(\rho, \mu) : \mu \in A\}$.

It is important to note whether $\mathcal{H}_\Psi$ is subordinate semimetric on the set $P_{cb}(\Gamma)$ or not. The next result demonstrates that $(P_{cb}(\Gamma), \mathcal{H}_\Psi)$ is a subordinate semimetric space.

Following [2], a condition is added to subordinate semimetric spaces in this section, which allows us to loosen the triangle inequality and provide several additional fixed point results on the family of SVMs. Assume Ψ is a subordinate semimetric space that meets the following condition: For every sequence $\{\rho_n\}, \{\mu_n\}$

$$\lim_{n \to \infty} \Psi(\rho_n, \mu_n) = 0 \text{ implies } \limsup_{n \to \infty} \Psi^*(\rho_n, A) = \limsup_{n \to \infty} \Psi^*(\mu_n, A). \quad (1)$$

Below we give two examples for which (1) holds and another one for which (1) does not hold.

Example 1. *Let $\Gamma = [0, 1]$. Let $\Psi : \Gamma \times \Gamma \to [0, \infty]$ given by*

$$\Psi(\omega, \nu) = \Psi(\nu, \omega) = \begin{cases} n^2, & \text{if } (\omega, \nu) = \left(\frac{1}{n}, 0\right), n \in \mathbb{N}; \\ n, & \text{if } (\omega, \nu) = \left(\frac{r}{n(r+1)}, 0\right), n, r \in \mathbb{N}; \\ (\omega - \nu)^2, & \text{otherewise.} \end{cases}$$

Let $m \in \mathbb{N}$.

Note that $\lim_{n\to\infty} \Psi\left(\frac{1}{m}, \frac{n}{m(n+1)}\right) = \lim_{n\to\infty} \left(\frac{1}{m} - \frac{n}{m(n+1)}\right)^2 = 0$, *and* $\lim_{n,r\to\infty} \Psi\left(\frac{n}{m(n+1)}, \frac{r}{m(r+1)}\right) = \lim_{n,r\to\infty} \left(\frac{n}{m(n+1)} - \frac{r}{m(r+1)}\right)^2 = 0.$

Thus the sequence $\left\{\frac{n}{m(n+1)}\right\}_{n\in\mathbb{N}}$ *is an infinite Cauchy sequence that is convergent to* $\frac{1}{m}$. *Now, suppose there is* $c > 0$ *such that*

$$m^2 = \Psi\left(\frac{1}{m}, 0\right) \leq \limsup_{n\to\infty} \Psi\left(\frac{n}{m(n+1)}, 0\right) = cm,$$

then $c \geq m$ *for all* $m \in \mathbb{N}$. *Hence* (Γ, Ψ) *is not RS space.*

Note that (Γ, Ψ) *is subordinate semimetric to* $\xi(t) = \begin{cases} t, & 0 \leq t \leq 1; \\ t^4, & t > 1. \end{cases}$

Let $\{\rho_n = \frac{1}{n}\}, \{\mu_n = \sin\left(\frac{1}{n}\right)\} \subset \Gamma$. *Let* $\Delta = \{0\} \subset \Gamma$.

Note that

$$\lim_{n\to\infty} \Psi(\rho_n, \mu_n) = \lim_{n\to\infty} \Psi\left(\frac{1}{n}, \sin\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)^2 = 0.$$

Now, $\Psi^*(\rho_n, A) = \inf\{\Psi(\frac{1}{n}, 0)\} = \Psi(\frac{1}{n}, 0) = n^2$, *hence* $\limsup_{n\to\infty} \Psi^*(\rho_n, \Delta) = \infty$.

Also, $\Psi^*(\mu_n, \Delta) = \inf\{\Psi(\sin\left(\frac{1}{n}\right), 0)\} = \Psi(\sin\left(\frac{1}{n}\right), 0) = \sin^2\left(\frac{1}{n}\right)$, *hence*

$$\limsup_{n\to\infty} \Psi^*(\mu_n, \Delta) = 0.$$

Thus the condition (1) does not hold.

Example 2. *Let* $\Gamma = [0, 1]$. *Let* $\Psi : \Gamma \times \Gamma \to [0, \infty]$ *given by*

$\Psi(\omega, \nu) = \Psi(\nu, \omega) = \begin{cases} 2, & \text{if } (\omega, \nu) = (1, 0); \\ |\omega - \nu|, & \text{otherewise.} \end{cases}$

Note that (Γ, Ψ) *is subordinate semimetric to* $\xi(t) = 2t$.

Let $\{\rho_n\}, \{\mu_n\} \subset \Gamma$. *Let* $\Delta \subset \Gamma$. *Assume that*

$$\lim_{n\to\infty} \Psi(\rho_n, \mu_n) = \lim_{n\to\infty} |\rho_n - \mu_n| = 0.$$

Now, for $\kappa \in \Delta$, *since* $||\rho_n - \kappa| - |\mu_n - \kappa|| \leq |\rho_n - \mu_n|$, *then*

$$\limsup_{n\to\infty} \Psi^*(\rho_n, \Delta) = \limsup_{n\to\infty} \Psi^*(\mu_n, \Delta).$$

Thus the condition (1) holds.

In the remainder of this paper, we will investigate subordinate semimetric spaces that satisfy (1). Throughout this paper, the following lemma is important.

Lemma 1. *Assume* (Γ, Ψ) *is a subordinate semimetric space that satisfies (1) and* $\{\rho_n\} \subseteq \Gamma$ *is a sequence that is convergent to* $\rho \in \Gamma$. *Then, for every* $\mu \in \Gamma$ *we obtain*

$$\limsup_{n\to\infty} \Psi(\rho_n, \mu) = \Psi(\rho, \mu).$$

Proof. Using the hypothesis, (Γ, Ψ) meets the condition (1) and $\lim_{n\to\infty} \Psi(\rho_n, \rho) = 0$. Taking into account $\{\mu_n = \rho\}$ for each $n \in N$ and $A = \{\mu\}$ in (1), we can deduce that

$$\limsup_{n\to\infty} \Psi(\rho_n, \mu) = \limsup_{n\to\infty} \Psi(\mu_n, \mu) = \Psi(\rho, \mu).$$

The addition of property (1) to subordinate semimetric space makes the process of reaching the limit more natural. □

Lemma 2. *Let* (Γ, Ψ) *be a subordinate semimetric space to* ξ *, and let* ξ *be continuous. Then* $(P_{cb}(\Gamma), \mathcal{H}_\Psi)$*, is a subordinate semimetric space.*

Proof. We must prove that the requirements in Definition 1 are met. Starting with (D1), assume that the subsets $A, B \in P_{cb}(\Gamma)$ with $\mathcal{H}_\Psi(A, B) = 0$. Thus,

$$\sup_{\rho \in A} \Psi^*(\rho, B) = \sup_{\rho \in B} \Psi^*(\rho, A) = 0.$$

Let $\rho \in A$ be arbitrary, hence $\Psi^*(\rho, B) = 0$. Then, for every $n \in N$ there is a convergent sequence $\{b_n\}$ in B which is convergent to ρ. Therefore, $\rho \in \bar{B} = B$. As a result, $A \subseteq B$. Using the similar argument in the other case, we have $A = B$. Hence, (D1) is proved. (D2) is clear to see. To show (D3), we have to prove that there exists a function $\xi : [0, \infty] \to [0, \infty]$ with (i) ξ is non-decreasing; (ii) $\lim_{\rho \to 0} \xi(\rho) = 0$, such that for $(A, B) \in P_{cb}(\Gamma) \times P_{cb}(\Gamma)$; $A \neq B$ and the infinite Cauchy sequence $\{A_n\}$ which is convergent to A, thus

$$\mathcal{H}_\Psi(A, B) \leq \xi(\limsup_{n \to \infty} \mathcal{H}_\Psi(A_n, B)).$$

Assume that $\rho \in B, \mu \in A$ are arbitrary. As $\lim_{n \to \infty} \mathcal{H}_\Psi(A_n, A) = 0$, then for every $n \in \mathbb{N}$, there is $k_n > 0$ such that for every $n \geq k_n$ we obtain

$$\mathcal{H}_\Psi(A_n, A) < \frac{1}{n}. \tag{2}$$

For a fixed and arbitrary n, using the concept of $\mathcal{H}_\Psi$, for any $\epsilon > 0$, then there is $\mu_\epsilon \in A_n$ such that,

$$\Psi(\mu, \mu_\epsilon) < \mathcal{H}_\Psi(A_n, A) + \epsilon. \tag{3}$$

Using both (2) and (3), we deduce that for every $n \geq k_n$,

$$\Psi(\mu, \mu_\epsilon) < \frac{1}{n} + \epsilon.$$

As a consequence, we can select a subsequence $\{\mu_m\} \subseteq A_n$ such that $\{\mu_m\}$ is an infinite Cauchy sequence that converges to μ. As (Γ, Ψ) is a subordinate semimetric space, so by (D3), there exists a function $\xi : [0, \infty] \to [0, \infty]$ where ξ is non-decreasing; $\lim_{\rho \to 0} \xi(\rho) = 0$, and

$$\Psi(\rho, \mu) \leq \xi(\limsup_{m \to \infty} \Psi(\rho, \mu_m)).$$

Then, we obtain

$$\Psi^*(\rho, A) \leq \Psi(\rho, \mu) \leq \xi(\limsup_{m \to \infty} \Psi(\rho, \mu_m)),$$

Also, since

$$\limsup_{m \to \infty} \Psi(\rho, \mu_m) = \inf_{k \geq 1}(\sup_{m \geq k} \Psi(\rho, \mu_m))$$

and

$$\inf_{k \geq 1}(\sup_{m \geq k} \Psi(\rho, \mu_m)) \leq \sup_{m \geq k} \Psi(\rho, \mu_m)$$

and ξ is nondecreasing, apply ξ to both sides of the above inequalities, we have

$$\xi(\limsup_{m \to \infty} \Psi(\rho, \mu_m)) = \xi(\inf(\sup \Psi(\rho, \mu_m)))$$

and

$$\xi(\inf_{k\geq 1}(\sup_{m\geq k}\Psi(\rho,\mu_n))) \leq \xi(\sup_{m\geq k}\Psi(\rho,\mu_m)).$$

Now, we have

$$\Psi^*(\rho,A) \leq \xi(\inf_{k\geq 1}(\sup_{m\geq k}\Psi(\rho,\mu_m))) \leq \xi(\sup_{m\geq k}\Psi(\rho,\mu_m)). \tag{4}$$

For every $k \in \mathbb{N}$, there is $m_k \geq k > 0$ with

$$\sup_{m\geq k}(\Psi(\rho,\mu_m)) < \Psi(\rho,\mu_{m_k}) + \epsilon. \tag{5}$$

since ξ is non-decreasing, apply ξ to (5) to obtain

$$\xi(\sup(\Psi(\rho,\mu_m)) < \xi(\Psi(\rho,\mu_{m_k}) + \epsilon), \tag{6}$$

When (4) and (5) are combined, one can deduce that

$$\Psi^*(\rho,A) \leq \xi(\Psi(\rho,\mu_{m_k}) + 2\epsilon \leq \xi(\Psi^*(\rho,A_n) + \epsilon) \leq \xi(\sup_{\rho\in B}\Psi^*(\rho,A_n) + \epsilon)),$$

since ξ is non-decreasing and $\Psi^*(\rho,A_n) + \epsilon \leq \sup(\Psi^*(\rho,A_n) + \epsilon)$.

When ϵ approaches zero, using the continuity of ξ the following results can be obtained:

$$\Psi^*(\rho,A) \leq \xi(\sup_{\rho\in B}\Psi^*(\rho,A_n)).$$

Hence,

$$\sup_{\rho\in B}\Psi^*(\rho,A) \leq \xi(\sup_{\rho\in B}\Psi^*(\rho,A_n)). \tag{7}$$

Applying lim sup to (7), we obtain

$$\limsup_{n\to\infty}(\sup_{\rho\in B}\Psi^*(\rho,A)) \leq \limsup_{n\to\infty}\xi((\sup_{\rho\in B}\Psi^*(\rho,A_n));$$

also, using the continuity of ξ, we have

$$\sup_{\rho\in B}\Psi^*(\rho,A) \leq \xi(\limsup_{n\to\infty}\sup_{\rho\in B}\Psi^*(\rho,A_n)). \tag{8}$$

Using the concept of $\mathcal{H}_\Psi$ and by (8), we obtain

$$\mathcal{H}_\Psi(A,B) \leq \xi(\limsup_{n\to\infty}\mathcal{H}_\Psi(A_n,B)).$$

Thus, (D3) is satisfied. □

Lemma 3. *Let (Γ,Ψ) be a subordinate semimetric space that meets (1), and assume $\{\rho_n\}$ is a sequence that is Ψ-convergent to ρ. Hence*

$$\limsup_{n\to\infty}\Psi^*(\rho_n,A) = \Psi^*(\rho,A), \textit{for every } A \subseteq \Gamma.$$

Proof. It is obvious because of the new condition (1) and the convergence of ρ_n to ρ in (Γ,Ψ). □

3. Main Results

We provide our fundamental result in this section. The following notation is required throughout this study. For every $\{\rho_n\} \subset \Gamma$ define

$$\delta(\Psi, T, \rho_n) = \sup\{\Psi(\rho_{i+1}, \rho_{j+1}) : \rho_{i+1} \in T\rho_i, \rho_{j+1} \in T\rho_j \text{ and } i,\, j \geq n\}.$$

Theorem 4. *Assume (Γ, Ψ) is a complete subordinate semimetric space and assume $T : \Gamma \to P_{cb}(\Gamma)$ is an SVM. Assume that there is $\alpha \in [0,1]$ such that for every $\rho, \mu \in \Gamma$*

$$\mathcal{H}_\Psi(T\rho, T\mu) \leq \alpha \Psi(\rho, \mu).$$

If there exists $\rho_0 \in \Gamma$ with $\delta(\Psi, T, \rho_0) < \infty$, then T possesses a fixed point.

Proof. Assume ρ_0 is an arbitrary element in Γ and $\rho_1 \in T\rho_0$. If $\rho_1 = \rho_0$, then ρ_0 is the fixed point of T. Now let $\rho_0 \neq \rho_1$. Hence, for $\epsilon = (\frac{1}{\sqrt{\alpha}} - 1)\mathcal{H}_\Psi(\rho_1, \rho_0)$, there is $\rho_2 \in T\rho_1$ with

$$\Psi(\rho_1, \rho_2) < \mathcal{H}_\Psi(T\rho_1, T\rho_0) + \epsilon = \frac{1}{\sqrt{\alpha}}\mathcal{H}_\Psi(T\rho_1, T\rho_0) \leq \sqrt{\alpha}\Psi(\rho_1, \rho_0).$$

Hence, for any given $\rho_n \in T\rho_{n-1}$, there exist $\rho_{n+1} \in T\rho_n$ such that

$$\Psi(\rho_{n+1}, \rho_n) \leq \sqrt{\alpha}\Psi(\rho_n, \rho_{n-1}).$$

Thus, taking $\beta = \sqrt{\alpha}$, one can deduce that

$$\begin{aligned} \Psi(\rho_{n+1}, \rho_n) &\leq \beta\Psi(\rho_n, \rho_{n-1}) \leq \beta^2\Psi(\rho_{n-1}, \rho_{n-2}) \\ &\leq \ldots \leq \beta^n\Psi(\rho_0, \rho_1) \leq \beta^n\delta(\Psi, T, \rho_0). \end{aligned} \tag{9}$$

As $\delta(\Psi, T, \rho_0) < \infty$ and $\beta \in (0,1)$ it provides that

$$\lim_{n\to\infty} \Psi(\rho_{n+1}, \rho_n) = 0.$$

Regarding (9), we have

$$\delta(\Psi, T, \rho_n) \leq \beta^n\delta(\Psi, T, \rho_0).$$

Then, for every $n, m \in \mathbb{N}$, we obtain

$$\Psi(\rho_n, \rho_{n+m}) \leq \delta(\Psi, T, \rho_n) \leq \beta^n\delta(\Psi, T, \rho_0).$$

Hence

$$\lim_{n,m\to\infty} \Psi(\rho_n, \rho_{n+m}) = 0.$$

Thus, $\{\rho_n\}$ is a Cauchy sequence and since Γ is complete, it is convergent to some $\sigma \in \Gamma$.

Now we prove that $\sigma \in T\sigma$.

$$\limsup_{n\to\infty} \Psi^*(\rho_{n+1}, T\sigma) \leq \limsup_{n\to\infty} \mathcal{H}_\Psi(T\rho_n, T\sigma) \leq \limsup_{n\to\infty} \alpha\Psi(\rho_n, \sigma) = 0.$$

Therefore, $\limsup\limits_{n\to\infty} \Psi^*(\rho_{n+1}, T\sigma) = 0$. Using Lemma (3), we obtain

$$\Psi^*(\sigma, T\sigma) = \limsup_{n\to\infty} \Psi^*(\rho_{n+1}, T\sigma) = 0.$$

Thus, it implies $\sigma \in T\sigma$ which completes the proof . □

4. Well-Behaved Quasi-Contraction

Let (Γ, ϱ) be a metric space. An SVM $T : \Gamma \to P_{cb}(\Gamma)$ is called QC if there is some $0 < \alpha < 1$ with

$$\mathcal{H}(T\rho, T\mu) \leq \alpha \max\{\varrho(\rho, \mu), \varrho(\rho, T\rho), \varrho(\mu, T\mu), \varrho(\rho, T\mu), \varrho(\mu, T\rho)\}$$

for every $\rho, \mu \in \Gamma$.

Let (Γ, ϱ) be a complete distance space and let $T : \Gamma \to P_{cb}(\Gamma)$ be an SVQC for $\frac{1}{2} < \alpha < 1$. Is there a fixed point for T? We provide an answer to this above problem in subordinate semimetric spaces under some local constraint.

Definition 6. *Assume (Γ, Ψ) is a subordinate semimetric space and $T : \Gamma \to P_{cb}(\Gamma)$ is an SVM. A sequence $\{\rho_n\}$ is said to be an iterative sequence based on ρ_0 if $\rho_0 \in \Gamma$ and for all $n \in \mathbb{N}$, $\rho_n \in T\rho_{n-1}$.*

Definition 7. *Assume (Γ, Ψ) is a subordinate semimetric space and assume $T : \Gamma \to P_{cb}(\Gamma)$ is a SVM. For every $\rho, \mu \in \Gamma$ let*

$$N_\Psi(\rho, \mu) = \max\{\Psi(\rho, \mu), \Psi^*(\rho, T\rho), \Psi^*(\mu, T\mu), \Psi^*(\rho, T\mu), \Psi^*(\mu, T\rho)\},$$

T is called a WBQC if there is $\alpha \in (0, 1)$ with

$$\mathcal{H}_\Psi(T\rho, T\mu) \leq \alpha N_\Psi(\rho, \mu).$$

Also, for every iterative sequence $\{\rho_n\}$ with $\rho_n \neq \rho_{n-1}$, there is a sequence $\{s_n\}$ where $s_n \geq 1$ for all n such that

(i) $\Psi^*(\rho_{n-1}, T\rho_n) \leq s_n \Psi(\rho_{n-1}, \rho_n)$,
(ii) $\limsup\limits_{n\to\infty} s_n < \frac{1}{\sqrt{\alpha}}$.

Below is an example of a non well-behaved quasi contraction map.

Example 3. *Let $\Gamma = [0, 1]$. Let $\Psi : \Gamma \times \Gamma \to [0, \infty]$ be given by $\Psi(\omega, \nu) = |\omega - \nu|^p$, where p is a positive real number not equal to 1.*

Then (Γ, Ψ) is subordinate semimetric space to $\xi(t) = t^p$, $t \in [0, \infty]$.
Let $T : \Gamma \to P_{cb}(\Gamma)$ given by $T\omega = \{\frac{\omega}{2}\}$
Let $\omega, \nu \in \Gamma$. Then $T\omega = \{\frac{\omega}{2}\}$ and $T\nu = \{\frac{\nu}{2}\}$.

$$\begin{aligned}
\Psi^*(\frac{\omega}{2}, T\nu) &= \Psi(\frac{\omega}{2}, \frac{\nu}{2}) = \frac{|\omega - \nu|^p}{2^p}, \\
\Psi^*(\frac{\nu}{2}, T\omega) &= \Psi(\frac{\omega}{2}, \frac{\nu}{2}) = \frac{|\omega - \nu|^p}{2^p}, \\
\mathcal{H}_\Psi(T\omega, T\nu) &= \frac{|\omega - \nu|^p}{2^p} = \frac{1}{2^p}\Psi(\omega, \nu) \\
&\leq \frac{1}{2^p} N_\Psi(\omega, \mu).
\end{aligned}$$

Now, let $\alpha = \frac{1}{2^p}$. Then $\alpha \in (0, 1)$.

Let $\rho_0 = 1 \in \Gamma$, $\rho_1 \in T\rho_0 = T1 = \frac{1}{2}$, $\rho_2 \in T\rho_1 = T\frac{1}{2} = \frac{1}{4} = \frac{1}{2^2}$, and $\rho_n = \frac{1}{2^n}$. Now, $\{\rho_n = \frac{1}{2^n}\}$ is an iterative sequence such that $\rho_n = \frac{1}{2^n} \neq \frac{1}{2^{n-1}} = \rho_{n-1}$. Suppose there exists a sequence $\{s_n\} \subset [1, \infty)$ such that $\Psi^*(\rho_{n-1}, T\rho_n) \leq s_n \Psi(\rho_{n-1}, \rho_n)$ and $\limsup\limits_{n\to\infty} \sqrt{\alpha} s_n < 1$. Now,

$\Psi^*(\rho_{n-1}, T\rho_n) = \Psi^*(\frac{1}{2^{n-1}}, \frac{1}{2^{n+1}}) = \Psi(\frac{1}{2^{n-1}}, \frac{1}{2^{n+1}}) = \left|\frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}\right|^p = \frac{3^p}{2^{p(n+1)}} = \frac{3^p}{2^p 2^{pn}}$. *Also,* $\Psi(\rho_{n-1}, \rho_n) = \Psi(\frac{1}{2^{n-1}}, \frac{1}{2^n}) = \left|\frac{1}{2^{n-1}} - \frac{1}{2^n}\right|^p = \frac{1}{2^{pn}}$. *Note that* $\sqrt{\alpha} = \sqrt{\frac{1}{2^p}} = \frac{1}{\sqrt{2}^p}$. *Hence*

$$\frac{3^p}{2^p 2^{pn}} = \Psi^*(\rho_{n-1}, T\rho_n) \leq s_n \Psi(\rho_{n-1}, \rho_n) = s_n \frac{1}{2^{pn}}.$$

Thus $\frac{3^p}{2^p} \leq s_n \Leftrightarrow \frac{3^p}{\sqrt{2}^p} = \frac{3^p}{2^p}\sqrt{\alpha} \leq \sqrt{\alpha} s_n$.

Therefore $\left(\frac{3}{\sqrt{2}}\right)^p \leq \limsup_{n\to\infty} \sqrt{\alpha} s_n < 1$.

Thus, $p \ln\left(\frac{3}{\sqrt{2}}\right) < \ln 1 = 0$. *There is a contradiction, since* $p > 0$ *and* $\ln\left(\frac{3}{\sqrt{2}}\right) \approx 0.75 > 0$. *Hence* T *is not well-behaved quasi-contraction.*

Theorem 5. *Assume* (Γ, Ψ) *is a complete subordinate semimetric space and assume* $T : \Gamma \to P_{cb}(\Gamma)$ *is a WBQC. Furthermore, assume that there is* $\rho_0 \in \Gamma$ *with* $\delta(\Psi, T, \rho_0) < \infty$. *Then* T *possesses at least one fixed point in* Γ.

Proof. Assume $\rho_0 \in \Gamma$ is an arbitrary element and $\rho_1 \in T\rho_0$. If $\rho_1 = \rho_0$ then ρ_0 is a fixed point of T. Then assume that $\rho_0 \neq \rho_1$. Then, for $\epsilon = (\frac{1}{\sqrt{\alpha}} - 1)\mathcal{H}_\Psi(T\rho_1, T\rho_0)$, there exists $\rho_2 \in T\rho_1$ such that

$$\Psi(\rho_1, \rho_2) < \mathcal{H}_\Psi(T\rho_1, T\rho_0) + \epsilon = \frac{1}{\sqrt{\alpha}}\mathcal{H}_\Psi(T\rho_1, T\rho_0) \leq \sqrt{\alpha} N_\Psi(\rho_1, \rho_0).$$

Therefore, for a given $\rho_n \in T\rho_{n-1}$, there exist $\rho_{n+1} \in T\rho_n$ such that

$$\Psi(\rho_{n+1}, \rho_n) \leq \sqrt{\alpha} N_\Psi(\rho_n, \rho_{n-1}).$$

Using (i) and (ii) in Definition (7), one can also deduce that

$$\begin{aligned}
\Psi(\rho_{n+1}, \rho_n) &\leq \sqrt{\alpha} N_\Psi(\rho_n, \rho_{n-1}) \\
&\leq \sqrt{\alpha} \max\{\Psi(\rho_n, \rho_{n-1}), \Psi(\rho_n, \rho_{n+1}), \Psi^*(\rho_{n-1}, T\rho_n)\} \\
&\leq \sqrt{\alpha} \max\{\Psi(\rho_n, \rho_{n-1}), \Psi(\rho_n, \rho_{n+1}), s_n \Psi(\rho_{n-1}, \rho_n)\} \\
&= \sqrt{\alpha} s_n \Psi(\rho_n, \rho_{n-1}) \leq (\sqrt{\alpha} s_n)^2 \Psi(\rho_{n-1}, \rho_{n-2}) \\
&\leq \ldots \leq (\sqrt{\alpha} s_n)^n \Psi(\rho_1, \rho_0) \leq (\sqrt{\alpha} s_n)^n \delta(\Psi, T, \rho_0).
\end{aligned} \tag{10}$$

As $\delta(\Psi, T, \rho_0) < \infty$, $\sqrt{\alpha} \in (0, 1)$, then using (ii) of Definition (7), we obtain

$$\limsup_{n\to\infty} \sqrt[n]{(\sqrt{\alpha} s_n)^n} = \limsup_{n\to\infty} \sqrt{\alpha} s_n < 1.$$

Then $\sum_{n=1}^{\infty} (\sqrt{\alpha} s_n)^n < \infty$. Hence, $\lim_{n\to\infty} (\sqrt{\alpha} s_n)^n = 0$. Therefore, we obtain

$$\lim_{n\to\infty} \Psi(\rho_n, \rho_{n+1}) = 0.$$

Using (10), it leads to

$$\delta(\Psi, T, \rho_n) \leq (\sqrt{\alpha} s_n)^n \delta(\Psi, T, \rho_0).$$

Then, for every $n, m \in \mathbb{N}$ we obtain

$$\Psi(\rho_n, \rho_{n+m}) \leq \delta(\Psi, T, \rho_n) \leq (\sqrt{\alpha} s_n)^n \delta(\Psi, T, \rho_0).$$

Therefore,

$$\lim_{n,m\to\infty} \Psi(\rho_n, \rho_{n+m}) = 0.$$

Thus, $\{\rho_n\}$ is a Cauchy sequence. Hence it converges by completeness of Γ to some $\sigma \in \Gamma$.

Next, we prove that $\sigma \in T\sigma$. Suppose that $\Psi^*(\sigma, T\sigma) > 0$, so

$$\begin{aligned}\Psi^*(\rho_{n+1}, T\sigma) &\leq \mathcal{H}_\Psi(T\rho_n, T\sigma)\\ &\leq \alpha N_\Psi(\rho_n, \sigma)\\ &= \alpha \max\{\Psi(\rho_n, \sigma), \Psi^*(\rho_n, T\rho_n), \Psi^*(\sigma, T\sigma), \Psi^*(\rho_n, T\sigma), \Psi^*(\sigma, T\rho_n)\}\\ &\leq \alpha \max\{\Psi(\rho_n, \sigma), \Psi(\rho_n, \rho_{n+1}), \Psi^*(\sigma, T\sigma), \Psi^*(\rho_n, T\sigma), \Psi(\sigma, \rho_{n+1})\}. \end{aligned} \tag{11}$$

Using Lemma (3) and applying the limit supremum to (11), it is easy to deduce that

$$\Psi^*(\sigma, T\sigma) = \limsup_{n\to\infty} \Psi^*(\rho_{n+1}, T\sigma) \leq \limsup_{n\to\infty} \alpha N_\Psi(\rho_n, \sigma) \leq \alpha\Psi^*(\sigma, T\sigma) = 0.$$

Then, it leads to $\alpha > 1$ and this is a contradiction. Thus, $\sigma \in \overline{T\sigma} = T\sigma$ and this implies that $\sigma \in \Gamma$ is a fixed point of a function T. □

Theorem 6. *Assume (Γ, ϱ) is a complete metric space. For $0 < \alpha < 1$ assume that $T : \Gamma \to P_{cb}(\Gamma)$ meets the following:*

$$\mathcal{H}(T\rho, T\mu) \leq \alpha \max\{\varrho(\rho, \mu), \varrho(\rho, T\rho), \varrho(\mu, T\mu), \varrho(\rho, T\mu), \varrho(\mu, T\rho)\} \text{ for all } \rho, \mu \in \Gamma.$$

Furthermore, assume for every iterative sequence $\{\rho_n\}$ with $\rho_n \neq \rho_{n-1}$, there is a sequence $\{s_n\}$, where $s_n \geq 1$ for all n such that

(i) $\varrho(\rho_{n-1}, T\rho) \leq s_n\varrho(\rho_{n-1}, \rho_n)$,

(ii) $\limsup_{n\to\infty} s_n < \frac{1}{\sqrt{\alpha}}$.

Then T possesses at least one fixed point in Γ.

Proof. From Theorem (5) and by the completeness of subordinate semimetric space (Γ, ϱ), and T is WBQC, the desired result can be concluded. □

The next examples demonstrate that the set of WBQCs is nonempty and Theorem (5) is meaningful.

Example 4. *Let $\xi : [0, \infty) \to [0, \infty)$ be defined as $\xi(\omega) = \omega^{\frac{1}{2}}$, Given $q_0 \in (0, 1)$ there is $t_0 = (\frac{q_0}{2})^2 \in (0, 1)$ such that $\xi(t_0) \geq \frac{t_0}{q_0}$.*

Let us consider the set $\Gamma = \{0\} \bigcup \mathbb{N}$ and let $\Psi : \Gamma \times \Gamma \to [0, \infty)$ defined by

$$\Psi(\rho, \mu) = \Psi(\mu, \rho) = \begin{cases} t_0^{\min\{\rho,\mu\}}, & \text{if } (\rho, \mu) \in \mathbb{N} \times \mathbb{N};\\ \xi(t_0^\mu), & \text{othrwise.}\end{cases}$$

By example (8) in [1], (Γ, Ψ) is subordinate semimetric with $\xi = \omega^{\frac{1}{2}}$.

Define $T : \Gamma \to P_{cb}(\Gamma)$ by $T0 = \{0, 1\}$ and let $T\rho = \{\rho + 1\}$ for $\rho \geq 1$, we want to show that

$$\mathcal{H}_\Psi(T\rho, T\mu) \leq \alpha N_\Psi(\rho, \mu).$$

Let $\rho, \mu \geq 1$ and without loss of generality we can suppose that $\rho > \mu$. Then,

$$\mathcal{H}_\Psi(T\rho, T\mu) = \max\{\sup_{\omega\in T\mu} \Psi^*(\omega, T\rho), \sup_{\omega\in T\rho} \Psi^*(\omega, T\mu)\},$$

in which, $\Psi^*(\omega, T\rho) = \inf\{\Psi(\omega, \nu) : \nu \in T\rho\}$ and $\Psi^*(\omega, T\mu) = \inf\{\Psi(\omega, \nu) : \nu \in T\mu\}$.

Now ,

$$\begin{aligned}\Psi^*(\omega, T\rho) &= \inf\{\Psi(\omega, \nu) : \nu \in T\rho\} \\ &= \Psi(\omega, \rho + 1) \\ \Psi^*(\omega, T\mu) &= \inf\{\Psi(\omega, \mu) : \mu \in T\mu\} \\ &= \Psi(\omega, \mu + 1).\end{aligned}$$

Then,

$$\mathcal{H}_\Psi(T\rho, T\mu) = \max\{\sup_{\omega \in T\mu} \Psi(\omega, \rho + 1), \sup_{\omega \in T\rho} \Psi(\omega, \mu + 1)\}.$$

Since $\rho \geq \mu$, $\rho, \mu \geq 1$, *then*

$$\begin{aligned}\mathcal{H}_\Psi(T\rho, T\mu) &= \max\{t_0^{\min\{\mu+1, \rho+1\}}, t_0^{\min\{\rho+1, \mu+1\}}\}, \\ &= \max\{t_0^{\mu+1}, t_0^{\mu+1}\} \\ &= t_0^{\mu+1}.\end{aligned}$$

Thus,

$$\mathcal{H}_\Psi(T\rho, T\mu) = t_0^{\mu+1}.$$

On the other hand,

$$\mathcal{H}_\Psi(T\rho, T0) = \max\{\sup_{\omega \in T0} \Psi^*(\omega, T\rho), \sup_{\omega \in T\rho} \Psi^*(\omega, T0)\},$$

where

$$\begin{aligned}\Psi^*(\omega, T\rho) &= \inf\{\Psi(\omega, \rho) : \rho \in T\rho\, ; \omega \in \{0, 1\}\} \\ &= \inf\{\Psi(0, \rho + 1), \Psi(1, \rho + 1)\} \\ &= \inf\{\xi(t_0^{\rho+1}), t_0^{\min\{\rho+1, 1\}}\} \\ &= \inf\{\xi(t_0^{\rho+1}), t_0\} \\ &= \inf\{(t_0^{\frac{\rho+1}{2}}), t_0\} = t_0^{\frac{\rho+1}{2}}. \\ \textit{So, } \sup_{\omega \in T0} \Psi^*(\omega, T\rho) &= \sup_{\omega \in T0} \{\inf\{t_0^{\frac{\rho+1}{2}}, t_0\}\} = t_0^{\frac{\rho+1}{2}}.\end{aligned}$$

Further,

$$\begin{aligned}\Psi^*(\omega, T0) &= \inf\{\Psi(\omega, \rho) : \rho \in T0 = \{0, 1\}\} \\ &= \inf\{\Psi(\omega, 0), \Psi(\omega, 1)\} \\ &= \inf\{\Psi(\rho + 1, 0), \Psi(\rho + 1, 1)\} \\ &= \inf\{\xi(t_0^{\rho+1}), t_0^{\min\{\rho+1, 1\}}\} \\ &= \inf\{\xi(t_0^{\rho+1}), t_0^1\} \\ &= \inf\{\xi(t_0^{\rho+1}), t_0\} \\ &= \inf\{t_0^{\frac{\rho+1}{2}}, t_0\} = t_0^{\frac{\rho+1}{2}}. \\ \textit{So, } \sup_{\omega \in T\rho} \Psi^*(\omega, T0) &= \sup_{\omega \in T\rho} \{\inf\{t_0^{\frac{\rho+1}{2}}, t_0\}\} = t_0^{\frac{\rho+1}{2}}.\end{aligned}$$

Hence,

$$\mathcal{H}_\Psi(T\rho, T0) = \max\{t_0^{\frac{\rho+1}{2}}, t_0^{\frac{\rho+1}{2}}\}.$$

Therefore,

$$\mathcal{H}_{\Psi}(T\rho, T0) = t_0^{\frac{\rho+1}{2}}\text{ , where } \rho \geq 1.$$

Now,

$$N_{\Psi}(\rho, \mu) = \max\{\Psi(\rho, \mu), \Psi^*(\rho, T\rho), \Psi^*(\mu, T\mu), \Psi^*(\rho, T\mu), \Psi^*(\mu, T\rho)\}.$$

Note that

$$\Psi(\rho, \mu) = t_0^{\min\{\rho,\mu\}} = t_0^{\mu}, \text{ as } \rho \geq \mu.$$

So,

$$\begin{aligned}
\Psi^*(\rho, T\rho) &= \inf\{\Psi(\rho, \nu) : \nu \in T\rho\ \} \\
&= \inf\{t_0^{\min\{\rho,\rho+1\}}\} \\
&= \inf\{t_0^{\rho}\} = t_0^{\rho}. \\
\text{Also, } \Psi^*(\mu, T\mu) &= \inf\{\Psi(\mu, \nu) : \nu \in T\mu\} \\
&= \inf\{t_0^{\min\{\mu,\mu+1\}}\} \\
&= \inf\{t_0^{\mu}\} = t_0^{\mu} \\
\text{and } \Psi^*(\rho, T\mu) &= \inf\{\Psi(\rho, \nu) : \nu \in T\mu\} \\
&= \inf\{t_0^{\min\{\rho,\mu+1\}}\}.
\end{aligned}$$

If $\rho \leq \mu + 1 \Rightarrow \rho = \mu + 1$, *then*

$$\Psi^*(\rho, T\mu) = \inf\{t_0^{\mu+1}\} = t_0^{\mu+1}.$$

If $\rho \geq \mu + 1$, *then*

$$\begin{aligned}
\Psi^*(\rho, T\mu) &= \inf\{t_0^{\mu+1}\} \\
&= t_0^{\mu+1} \text{ and ,} \\
\Psi^*(\mu, T\rho) &= \inf\{\Psi(\mu, \nu) : \nu \in T\rho\} \\
&= \inf\{t_0^{\min\{\mu,\rho+1\}}\} \\
&= \inf\ \{t_0^{\mu}\} \\
&= t_0^{\mu} \text{ as } \rho \geq \mu.
\end{aligned}$$

Then,

$$\begin{aligned}
N_{\Psi}(\rho, \mu) &= \max\{t_0^{\mu}, t_0^{\rho}, t_0^{\mu}, t_0^{\mu+1}, t_0^{\mu}\} \\
N_{\Psi}(\rho, \mu) &= \max\{t_0^{\mu}, t_0^{\rho}, t_0^{\mu+1}\} = t_0^{\mu}. \\
N_{\Psi}(\rho, 0) &= \max\{\Psi(\rho, 0), \Psi^*(\rho, T\rho), \Psi^*(0, T0), \Psi^*(\rho, T0), \Psi^*(0, T\rho)\} \\
N_{\Psi}(\rho, 0) &= \max\{\xi(t_0^{\rho}), t_0^{\min\{\rho,\rho+1\}}, \inf\{\Psi(0, 0), \Psi(0, 1)\}, \inf\{\Psi(\rho, 0), \Psi(\rho, 1)\}, \xi(t_0^{\rho+1})\} \\
&= \max\{\xi(t_0^{\rho}), t_0^{\rho}, \inf\{1, \xi(t_0)\}, \inf\{\xi(t_0^{\rho}), t_0^{\min\{\rho,1\}}, \xi(t_0^{\rho+1})\} \\
&= \max\{t_0^{\frac{\rho}{2}}, t_0^{\rho}, \inf\{1, t_0^{\frac{1}{2}}\}, \inf\{t_0^{\frac{\rho}{2}}, t_0\}, t_0^{\frac{\rho+1}{2}}\} \\
&= \max\{t_0^{\frac{\rho}{2}}, t_0^{\rho}, t_0^{\frac{1}{2}}, t_0^{\frac{\rho}{2}}, t_0^{\frac{\rho+1}{2}}\} = t_0^{\frac{1}{2}}.
\end{aligned}$$

Note that since $\rho \geq \mu$, *we obtain*

$$\frac{\mathcal{H}_{\Psi}(T\rho, T\mu)}{N_{\Psi}(\rho, \mu)} = \frac{t_0^{(\mu+1)}}{t_0^{\mu}} = t_0^{\mu-\mu+1} = t_0,$$

then,

$$\mathcal{H}_\Psi(T\rho, T\mu) \leq t_0 N_\Psi(\rho,\mu), \text{ and}$$

$$\frac{\mathcal{H}_\Psi(T\rho, T0)}{N_\Psi(\rho,0)} = \frac{t_0^{\frac{\rho+1}{2}}}{t_0^{\frac{1}{2}}} = t_0^{\frac{\rho}{2}} \leq t_0, \text{ as } \rho \geq 1.$$

So, $\mathcal{H}_\Psi(T\rho, T\mu) \leq t_0 N_\Psi(\rho,\mu)$.

Moreover, there is only one iterative sequence $\{\rho_n\}$ *that has the initial element* $\rho_0 \in \Gamma$ *with* $\rho_n \neq \rho_{n-1}$ *which is given by* $\rho_{n+1} = \rho_n + 1$*, started with* ρ_0 *and since*

$$\Psi(0,\rho_n) = \xi(t_0^{\rho_n}) = t_0^{\frac{\rho_n}{2}}$$

then $\{\rho_n\}$ *converges to zero. let* $s_n = 1$ *for all* n , $\{s_n\} \subset [1,\infty]$

(i) $\Psi^*(\rho_{n-1}, T\rho_n) \leq s_n \Psi(\rho_{n-1},\rho_n)$ *because*

$$\begin{aligned} L.H.S &= \Psi^*(\rho_{n-1}, T\rho_n) = \inf\{\Psi(\rho_{n-1},\rho_{n+1})\} \\ &= t_0^{\min\{\rho_{n-1},\rho_{n+1}\}} \\ &= t_0^{\rho_{n-1}} \\ R.H.S &= s_n\Psi(\rho_{n-1},\rho_n) \\ &= t_0^{\min\{\rho_{n-1},\rho_n\}} \\ &= t_0^{\rho_{n-1}}. \end{aligned}$$

(ii) $\limsup_{n\to\infty} s_n = \limsup 1 = 1 \leq \frac{1}{\sqrt{t_0}}$

since $t_0 < 1$ *then* $\sqrt{t_0} < 1$ *then* $\frac{1}{\sqrt{t_0}} > 1$.

Hence, requirements (i) and (ii) of Definition (7) hold. Also, the sequence $\{\rho_n\}$ *converges to zero which is the fixed point of* T*. Therefore, Theorem (5) is satisfied.*

Example 5. *Let* $\Gamma = \{0\} \cup [2,\infty)$ *and let*

$$\Psi(\rho,\mu) = \begin{cases} \rho+\mu, & \text{if } \rho \neq 0 \text{ and } \mu \neq 0; \\ \frac{\rho}{2}, & \text{if } \mu = 0; \\ \frac{\mu}{2}, & \text{if } \rho = 0. \end{cases}$$

Let $T0 = \{0,2\}$ *and let* $T\rho = \{\frac{1}{1+\rho} + 2\}$ *for each* $\rho \geq 2$*. We show that* (Γ,Ψ) *is a subordinate semimetric space and*

$$\mathcal{H}_\Psi(T\rho, T\mu) \leq \frac{7}{26} N_\Psi(\rho,\mu).$$

Using Theorem (5), we can conclude that T *possesses a fixed point. To prove that* Ψ *is a subordinate semi-metric on* Γ,

- *(D1) for each pair* $(\rho,\mu) \in \Gamma^2$*, we have to prove that* $\Psi(\rho,\mu) = 0$ *implying that* $\rho = \mu$*, if* $\rho \neq 0$ *and* $\mu \neq 0$*; then,* $\Psi(\rho,\mu) = \rho+\mu = 0$ *is impossible because* ρ *and* μ *belong to* $[2,\infty]$ *and are non-negative. If* $\rho \neq 0$ *and* $\mu = 0$*, then* $\Psi(\rho,\mu) = \frac{\rho}{2} = 0$, $\rho = 0$ *and* $\rho = \mu$*. The other case is similar.*
- *(D2) for each pair* $(\rho,\mu) \in \Gamma^2$*, clearly we have* $\Psi(\rho,\mu) = \Psi(\mu,\rho)$.
- *(D3) let a function* $\xi : [0,\infty] \to [0,\infty]$ *be defined by* $\xi(\rho) = \rho^\varepsilon$*, where* $\varepsilon > 1$*, thus* ξ *is non-decreasing;* $\lim_{\rho\to 0} \xi(\rho) = 0$*. In part two of definition of* Ψ *we cannot find an infinite Cauchy sequence in* Γ *such that* (ρ_n) *is convergent to* 0.

In part one of definition of Ψ, assume $(\rho,\mu) \in \Gamma^2$, with $\rho \neq \mu$, and (ρ_n) is an infinite Cauchy sequence in Γ with (ρ_n) convergent to ρ, we obtain

$$\begin{aligned}\Psi(\rho,\mu) &= \Psi(\lim_{n\to\infty} \rho_n, \mu) \\ &= \lim_{n\to\infty} (\rho_n + \mu) \\ &= \limsup_{n\to\infty} (\rho_n + \mu) \\ &= \limsup_{n\to\infty} \Psi(\rho_n, \mu) \\ &\leq (\limsup_{n\to\infty} \Psi(\rho_n,\mu))^{\varepsilon} \\ &= \xi(\limsup_{n\to\infty} \Psi(\rho_n,\mu)).\end{aligned}$$

Therefore, (Γ,Ψ) is a subordinate semimetric space and is also clearly complete. Now, let $\rho,\mu \geq 2$ and without loss of generality we can suppose that $\rho > \mu$. Thus,

$$\mathcal{H}_\Psi(T\rho, T\mu) = \frac{\rho+\mu+2}{(\rho+1)(\mu+1)} \text{ and } \mathcal{H}_\Psi(T\rho, T0) = \frac{1}{2(\rho+1)} + 1.$$

Also,

$$N_\Psi(\rho,\mu) = \max\{\rho+\mu, \rho + \frac{1}{(1+\mu)} + 2\} \text{ and } N_\Psi(\rho,0) = \frac{1}{1+\rho} + 2 + \rho.$$

Note that, since $\rho > \mu$, we have

$$\begin{aligned}\frac{\mathcal{H}_\Psi(T\rho,T\mu)}{N_\Psi(\rho,\mu)} &= \frac{\frac{\rho+\mu+2}{(\rho+1)(\mu+1)}}{\max\{\rho+\mu, \rho+\frac{1}{(1+\mu)}+2\}} \\ &\leq \frac{1}{\max\{\rho+\mu, \frac{\rho+1}{(\mu+1)}+2\}} \\ &< \frac{1}{2\mu} \leq \frac{1}{4},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{\mathcal{H}_\Psi(T\rho,T0)}{N_\Psi(\rho,0)} &= \frac{\frac{1}{2(1+\rho)}+1}{\frac{1}{(\rho+1)}+2+\rho} = \frac{2\rho+3}{2\rho^2+6\rho+6} \\ &\leq \max_{\rho\geq 2}\{\frac{2\rho+3}{2\rho^2+6\rho+6}\} \\ &= \frac{7}{26}.\end{aligned} \tag{13}$$

Thus, (12) and (13) prompt us to choose $\alpha = \frac{7}{26}$ and hence

$$\mathcal{H}_\Psi(T\rho,T\mu) \leq \frac{7}{26} N_\Psi(\rho,\mu).$$

Moreover, there is only one iterative sequence $\{\rho_n\}$ that has the initial element $\rho_0 \in \Gamma$ with $\rho_n \neq \rho_{n-1}$ which is $\rho_{n+1} = \frac{1}{1+\rho_n} + 2$ starting with ρ_0. By Theorem (5) , $\{\rho_n\}$ is convergent to $\frac{1+\sqrt{13}}{2}$ and it is the fixed point of T. Let

$$s_n = \frac{\rho_{n-1} + \rho_{n+1}}{\rho_{n-1} + \rho_n},$$

we observe that (i) and (ii) of Definition (7) hold.

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