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Eigenvalue of (p,q)-Biharmonic System along the Ricci Flow

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Abstract: In this paper, we determine the variation formula for the first eigenvalue of (p,q)-biharmonic system on a closed Riemannian manifold. Several monotonic quantities are also derived.

Keywords: eigenvalue; p-biharmonic operator; Ricci flow; monotonicity

MSC: 53C21; 53C44

1. Introduction

The study of eigenvalues of geometric operators plays a major role in ascertaining the geometrical and topological properties of the underlying manifolds. This area of research became attractive after Perelman's work [1]. He presented a functional $\mathcal{F} = \int_M (R + |\nabla f|^2)e^{-f} \mathrm{d}\mu$. It is shown that this functional is nondecreasing under the Ricci flow with a backward heat-type equation. It explains that the first eigenvalue of $-4\Delta + R$ (R is the scalar curvature) is nondecreasing under the Ricci flow. Later, many authors studied properties of eigenvalues for different geometric operators on evolving Riemannian manifolds and other manifolds. For example, Cao, in [2], studied eigenvalues of $(-\triangle + \frac{R}{2})$ and obtained that the eigenvalues of $(-\triangle + \frac{R}{2})$ are nondecreasing under the Ricci flow for manifolds with a non-negative curvature operator. In Ref. [3], he considered the first eigenvalues of geometric operators under the Ricci flow. Bracken and Azami obtained some results on the evolution of the first eigenvalue recently [4–6]. Other remarkable work can be found in [7–10]. We call the smooth function $u : M \to \mathbb{R}$ a harmonic function if $\Delta u = 0$. Harmonic functions play a significant role in Dirichlet boundary value and Neumann boundary value problems. The function u is called biharmonic if $\Delta^2 u = 0$. Here, Δ^2 is known as the biharmonic operator. The biharmonic function has application to continuum mechanics and elasticity theory. It is known that every harmonic function is biharmonic, but the converse is not true. As a generalization, the function u is called p-biharmonic if $\Delta_p^2 u = 0$ for $p \in (1, +\infty)$, where Δ_p^2 is known as the p-biharmonic operator (an elliptic operator of fourth order), defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$. For $p = 2$, the p-biharmonic operator reduces to a harmonic operator. In Ref. [11], Khalil et al. researched the spectrum for the p-biharmonic operator. It was proved that the spectrum of the p-biharmonic operator with Dirichlet boundary conditions and indefinite weight includes at least one nondecreasing sequence of positive eigenvalues. The spectra of the Neumann p-biharmonic and Dirichlet problems were considered in [12]. In the last decade, there are many results on the p-biharmonic operator that can give us motivation. For example, Benedikt and Drábek studied the principal eigenvalue of the p-biharmonic operator in [13,14]. In Refs. [15–18], Khalil et al. researched the singular- and double-eigenvalue problems for the p-biharmonic operator. Work regarding the boundary value problems of the p-biharmonic operator can be found in [19–21]. Other properties of the p-biharmonic operator from different viewpoints can be seen in [22–25]. Recently,

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in [26] Abolarinwa et al. studied the demeanor of the spectrum of the p-biharmonic operator on a closed complete Riemannian manifold when the manifold evolves by the Ricci flow and found evolution formulas and some monotonic formulas along the Ricci flow. Abolarinwa [27] extended this to volume-preserving Ricci flow and found some useful applications on closed surfaces within restrictions on Euler characteristics, and also, on locally homogeneous 3-manifolds in Bianchi classes. Furthermore, in recent years Khan and Li et al. obtained some interesting results relevant to soliton theory [28], submanifold theory [29], singularity theory [30], classical differential geometry [31–33], and tangent bundle problems [34–38], etc. These papers give our inspiration and motivation in the present and future research work. In future work, we are going to find more new results combined with the techniques and the results in those papers. In Ref. [39], Azami studied the first eigenvalue of the $\Delta_p^2 - \Delta_p$, where Δ_p is the p-Laplace operator, on a closed Riemannian manifold along the Ricci flow, and by the application of some geometric conditions it was proved that the first eigenvalue is nondecreasing under the Ricci flow. In Ref. [40], Li and Tang established the existence of more than three solutions to a Navier boundary problem adhering to the (p,q)-biharmonic systems. Recently, in [41] the authors considered a system called a (p,q)-biharmonic system, given by

$$\begin{aligned}
&\Delta_p^2 u = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u, \text{ in } \Omega,\\
&\Delta_q^2 v = \lambda b(x)|v|^{q-2}v + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v, \text{ in } \Omega,\\
&u = \Delta u = 0, \ v = \Delta v = 0, \text{ on } \partial\Omega.
\end{aligned}$$

Here, $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a connected set and bounded; $\lambda > 0$ is a parameter; $p > 1$, $q > 1$, $\max\{p,q\} < \frac{N}{2}$, $\alpha \geq 0$, and $\beta \geq 0$; a, b, and c are positive functions defined in Ω and $c \neq 0$ in Ω.

Fourth-order PDEs are very useful in different fields of science, such as engineering [42–44], signal processing [45–48], nuclear physics [49,50], etc. As an example, these kind of PDEs arise in traveling waves of suspension bridges. To study the mechanical vibrations of plates, the eigenvalue problems of biharmonic systems plays an important role. Interested readers are directed to [41] for additional information.

Consider $d\mu$ as the Riemann volume measure of an n-dimensional closed Riemannian manifold (M^n, g). We now consider the (p,q)-biharmonic system below:

$$\begin{aligned}
&\Delta_p^2 u = \lambda |u|^{\alpha}|v|^{\beta}v, \text{ in } M,\\
&\Delta_q^2 v = \lambda |u|^{\alpha}|v|^{\beta}u, \text{ in } M,\\
&(u,v) \in W^{2,p}(M) \times W^{2,q}(M),
\end{aligned} \tag{1}$$

where $\alpha > 0$, $\beta > 0$, $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, and $W^{2,p}(M)$ is the Sobolev space, and study its eigenvalue on (M^n, g), evolving by the Ricci flow. Mainly under the normalized Ricci flow and Ricci flow, the variational formula for the first eigenvalue of the system (1) is derived. Along the Ricci flow we also deduce certain monotonic quantities.

2. Preliminaries

In this section, we first present the eigenvalue of the (p,q)-biharmonic operator and recall some standard evolution equations.

Definition 1. *λ is said to be an eigenvalue of the system (1) if $\exists$ is a pair of functions $(u,v) \in W^{2,p}(M) \times W^{2,q}(M)$; $u \neq 0$, $v \neq 0$ such that*

$$\begin{aligned}
&\int_M |\Delta u|^{p-2}\Delta u \Delta\phi d\mu = \lambda \int_M |u|^{\alpha}|v|^{\beta}\phi v d\mu,\\
&\int_M |\Delta v|^{q-2}\Delta v \Delta\psi d\mu = \lambda \int_M |u|^{\alpha}|v|^{\beta}u\psi d\mu
\end{aligned} \tag{2}$$

holds. Here, $(\phi, \psi) \in (W_0^{1,p}(M) \cap W^{2,p}(M)) \times (W_0^{1,q}(M) \cap W^{2,q}(M)) = E$ *and* $W_0^{1,p}(M)$ *denotes the closure of the set* $C_0^\infty(M)$ *in the space* $W^{1,p}(M)$*. The pair* (u, v) *is termed an eigenfunction corresponding to the eigenvalue* λ*.*

Let us consider the following two functionals:

$$A(u, v) = \frac{\alpha + 1}{p} \int_M |\Delta u|^p \mathrm{d}\mu + \frac{\beta + 1}{q} \int_M |\Delta v|^q \mathrm{d}\mu, \tag{3}$$

and

$$B(u, v) = \int_M |u|^\alpha |v|^\beta uv \mathrm{d}\mu. \tag{4}$$

The first positive eigenvalue of (1) is characterized by

$$\lambda(u, v) = inf\{A(u, v) : (u, v) \in E,\ B(u, v) = 1\}. \tag{5}$$

A one-parameter family of metrics $g(t)$ is said to satisfy the Ricci flow [51] if the equation below holds:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{6}$$

with $g(0) = g_0$. Here, R_{ij} stands for the Ricci tensor. Hamilton [51] showed the existence of solutions of the Ricci flow and also proved its uniqueness.

If $(M^n, g(t))$ is a solution of the Ricci flow (6) on the closed manifold (M^n, g_0), then

$$\lambda(t, u(t), v(t)) = \frac{\alpha + 1}{p} \int_M |\Delta u|^p \mathrm{d}\mu + \frac{\beta + 1}{q} \int_M |\Delta v|^q \mathrm{d}\mu, \tag{7}$$

is the evolution of the first eigenvalue of (1), where $(u(t), v(t))$ is a normalized eigenfunction, i.e., $B(u, v) = 1$ is associated with the eigenvalue λ.

The next lemma contains some evolution equations which are standard in the theory of Ricci flow. Their proofs are omitted here but an interested reader can consult Ref. [52].

Lemma 1. *Under Ricci flow, the following equations hold:*

$$(1)\ \frac{\partial}{\partial t}(\mathrm{d}\mu) = -R\mathrm{d}\mu \qquad (2)\ \frac{\partial}{\partial t}\Delta u = 2R^{ij}\nabla_i \nabla_j u + \Delta u_t$$

$$(3)\ \frac{\partial}{\partial t} g^{ij} = 2R^{ij} \qquad (4)\ \frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2$$

where g^{ij} *is the inverse matrix of* g_{ij}*,* $\mathrm{d}\mu$ *is the volume element,* Δ *is the Laplace operator, and* R *is the scalar curvature.*

3. Variation Formula

Before continuing, we note that, as far as we are aware, it remains uncertain if the first eigenvalue of the system (1) or its associated eigenfunctions possess C^1-differentiability along the Ricci flow.

Then, for the procedure we adopt in this paper we need to introduce smooth functions defined at $t_0 \in [0, T)$ and a continuous eigenvalue.

Lemma 2. *For a given time* $t_0 \in [0, T)$*, there exists* C^∞ *functions* $u(t_0)$ *and* $v(t_0)$ *satisfying*

$$\int_M |u|^\alpha |v|^\beta uv \mathrm{d}\mu_{g(t)} = 1,$$

such that at t_0*,* $(u_0, v_0) = (u(t_0), v(t_0))$ *is the eigenfunction corresponding to* $\lambda_1(t_0)$*. Here,* λ_1 *denotes the first eigenvalue of the system* (1)*.*

Proof. At time t_0, set $(u_0, v_0) = (u(t_0), v(t_0))$ to be the eigenfunction corresponding to $\lambda_1(t_0)$. Consider the following smooth functions along the Ricci flow:

$$h(t) = u_0 \left(\frac{\det(g(t_0))}{\det(g(t))} \right)^{\frac{1}{2(\alpha+\beta+1)}} \quad \text{and} \quad w(t) = v_0 \left(\frac{\det(g(t_0))}{\det(g(t))} \right)^{\frac{1}{2(\alpha+\beta+1)}}.$$

Moreover, these functions can be normalized as follows:

$$u(t) = \frac{h(t)}{\left(\int_M |h(t)|^\alpha |w(t)|^\beta h(t) w(t) \mathrm{d}\mu_{g(t)} \right)^{\frac{1}{p}}}$$

and

$$v(t) = \frac{w(t)}{\left(\int_M |h(t)|^\alpha |w(t)|^\beta h(t) w(t) \mathrm{d}\mu_{g(t)} \right)^{\frac{1}{q}}}.$$

Clearly, the above functions $u(t)$ and $v(t)$ are smooth along the Ricci flow and can be shown to have satisfied the condition

$$\int_M |u|^\alpha |v|^\beta uv \mathrm{d}\mu_{g(t)} = 1.$$

□

Proposition 1. *Suppose $(M^n, g(t))$ is a solution of (6). For any $t_1, t_2 \in [0, T)$ (with t_1 being sufficiently close to t_2), $\epsilon > 0$, and $g(t_1)$ and $g(t_2)$ satisfying*

$$(1+\epsilon)^{-1} g(t_1) < g(t_2) < (1+\epsilon) g(t_1),$$

we have

$$\lambda(g(t_2)) - \lambda(g(t_1)) \leq ((1+\epsilon)^{\frac{p+n}{2}} - (1+\epsilon)^{\frac{n}{2}}) \lambda(g(t_1)),$$

for $p \geq q > 1$. In particular, $\lambda(t)$ is continuous.

The above lemma is an adaptation of lemma 3.1 in [53] and their proofs are similar, so the proof of Proposition 1 is omitted.

Proposition 2. *Suppose $(M^n, g(t))$ is a solution of (6). Let $\lambda_1(t)$ be the first eigenvalue of the (p, q)-biharmonic system (1) under the flow. For any $t_0, t_1 \in [0, T)$ such that $t_0 < t_1$, we have*

$$\lambda_1(t_1) \geq \lambda_1(t_0) + \int_{t_0}^{t_1} \mathcal{G}(t) dt,$$

where

$$\begin{aligned} \mathcal{G}(t) :=& 2(\alpha+1) \int_M |\Delta u|^{p-2} \Delta u (R^{ij} \nabla_i \nabla_j u + \frac{1}{2} \Delta u_t) \mathrm{d}\mu_{g(t)} \\ &+ 2(\beta+1) \int_M |\Delta v|^{q-2} \Delta v (R^{ij} \nabla_i \nabla_j v + \frac{1}{2} \Delta v_t) \mathrm{d}\mu_{g(t)} \\ &- \frac{\alpha+1}{p} \int_M R |\Delta u|^p \mathrm{d}\mu_{g(t)} - \frac{\beta+1}{q} \int_M R |\Delta v|^q \mathrm{d}\mu_{g(t)}. \end{aligned}$$

Proof. By definition

$$\lambda(t) = \frac{\alpha+1}{p} \int_M |\Delta u(t)|^p \mathrm{d}\mu_{g(t)} + \frac{\beta+1}{q} \int_M |\Delta v(t)|^p \mathrm{d}\mu_{g(t)}.$$

Using the functions $u(t)$ and $v(t)$ defined in Lemma 2, the time derivative of $\lambda(t)$ under the Ricci flow (6) yields

$$\frac{d\lambda(t)}{dt} = \frac{\partial}{\partial t}\left(\frac{\alpha+1}{p}\int_M |\Delta u(t)|^p \mathrm{d}\mu_{g(t)} + \frac{\beta+1}{q}\int_M |\Delta v(t)|^p \mathrm{d}\mu_{g(t)}\right) =: \mathcal{G}(t). \tag{8}$$

Applying Lemma 1 and the evolution $(|\Delta u(t)|^p)_t$ (see (12) below), it then follows that

$$\begin{aligned}\mathcal{G}(t) =& 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M R|\Delta u|^p \mathrm{d}\mu \\ &+ 2(\beta+1)\int_M |\Delta v|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v \mathrm{d}\mu - \frac{\beta+1}{q}\int_M R|\Delta v|^q \mathrm{d}\mu \\ &+ (\alpha+1)\int_M |\Delta u|^{p-2}\Delta u \Delta u_t \mathrm{d}\mu \\ &+ (\beta+1)\int_M |\Delta v|^{q-2}\Delta v \Delta v_t \mathrm{d}\mu,\end{aligned}$$

which is the same as $\mathcal{G}$ defined in the statement of the proposition (see the detailed computation in (13) below). Integrating both sides of (8) with respect to t on $[t_0, t_1]$, we obtain

$$\lambda(t_1) - \lambda(t_0) = \int_{t_0}^{t_1} \mathcal{G}(t)dt.$$

Since in this case $\lambda(t_1) = \lambda_1(t_1)$ and $\lambda(t_0) \geq \lambda_1(t_0)$, we arrive at

$$\lambda_1(t_1) \geq \lambda_1(t_0) + \int_{t_0}^{t_1} \mathcal{G}(t)dt. \tag{9}$$

□

Remark 1. *The above inequality (9) can be used to establish the monotonicity and differentiability of $\lambda_1(t)$ in the flow interval $[0, T)$. Here, if it can be established that $\int_{t_0}^{t_1} \mathcal{G}(t)dt > 0$ in any small neighborhood of t_1, then we have*

$$\lambda_1(t_1) > \lambda_1(t_0),$$

for any $t_0 < t_1$, with t_0 being sufficiently close to t_1. Two conclusions can easily be drawn as follows: (1) Since $t \in (0, T)$ is arbitrary, then $\lambda_1(t)$ is strictly increasing on $[0, T)$; and (2) invoking the classical Lebesgue theorem, based on the monotonicity and continuity properties, $\lambda_1(t)$ is almost everywhere differentiable along $g(t), t \in [0, T)$.

3.1. Variation in Eigenvalue along the Unnormalized Ricci Flow

We now introduce a new quantity

$$\lambda(t, u(t), v(t)) := \frac{\alpha+1}{p}\int_M |\Delta u(t)|^p \mathrm{d}\mu_{g(t)} + \frac{\beta+1}{q}\int_M |\Delta v(t)|^p \mathrm{d}\mu_{g(t)},$$

where $u(t)$ and $v(t)$ are smooth functions satisfying the normalization condition

$$\int_M |u|^\alpha |v|^\beta uv \mathrm{d}\mu_{g(t)} = 1.$$

The function $\lambda(t, u(t), v(t))$ is a smooth function with respect to the variable t. If (u, v) are the corresponding eigenfunctions of $\lambda(t_0)$, then $\lambda(t_0, u(t_0), v(t_0)) = \lambda(t_0)$. In general, $\lambda(t, u, v) \neq \lambda(t)$ but are equal at $t = t_0$.

The variation formula for the eigenvalue $\lambda(t)$ along the Ricci flow is therefore as follows:

Proposition 3. *Suppose $(M^n, g(t))$ is a solution of (6) on the smooth closed manifold (M^n, g_0). If λ denotes the evolution of the first eigenvalue of the (p,q)-biharmonic system (1), then*

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} = {} & 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M R|\Delta u|^p \mathrm{d}\mu \\ & + 2(\beta+1)\int_M |\Delta v|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v \mathrm{d}\mu - \frac{\beta+1}{q}\int_M R|\Delta v|^q \mathrm{d}\mu \\ & + \lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv \mathrm{d}\mu, \end{aligned} \tag{10}$$

where $(u(t), v(t))$ is a normalized eigenfunction associated with the eigenvalue $\lambda(t)$.

Proof. Differentiating (7) with respect to t at $t = t_0$, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} = {} & \frac{\alpha+1}{p}\int_M \frac{\partial}{\partial t}(|\Delta u|^p)\mathrm{d}\mu + \frac{\beta+1}{q}\int_M \frac{\partial}{\partial t}(|\Delta v|^q)\mathrm{d}\mu \\ & - \frac{\alpha+1}{p}\int_M R|\Delta u|^p \mathrm{d}\mu - \frac{\beta+1}{q}\int_M R|\Delta v|^q \mathrm{d}\mu. \end{aligned} \tag{11}$$

We have

$$\frac{\partial}{\partial t}(|\Delta u|^p) = p|\Delta u|^{p-2}\Delta u\{2R^{ij}\nabla_i\nabla_j u + \Delta u_t\}. \tag{12}$$

Thus, from (11), we obtain

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} = {} & 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M R|\Delta u|^p \mathrm{d}\mu \\ & + 2(\beta+1)\int_M |\Delta v|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v \mathrm{d}\mu - \frac{\beta+1}{q}\int_M R|\Delta v|^q \mathrm{d}\mu \\ & + (\alpha+1)\int_M |\Delta u|^{p-2}\Delta u \Delta u_t \mathrm{d}\mu \\ & + (\beta+1)\int_M |\Delta v|^{q-2}\Delta v \Delta v_t \mathrm{d}\mu. \end{aligned} \tag{13}$$

From

$$\int_M |u|^\alpha|v|^\beta uv \mathrm{d}\mu = 1$$

we obtain

$$(\alpha+1)\int_M |u|^\alpha|v|^\beta v u_t \mathrm{d}\mu + (\beta+1)\int_M |u|^\alpha|v|^\beta u v_t \mathrm{d}\mu = \int_M R|u|^\alpha|v|^\beta uv \mathrm{d}\mu. \tag{14}$$

So,

$$\begin{aligned} & (\alpha+1)\int_M |\Delta u|^{p-2}\Delta u\Delta u_t \mathrm{d}\mu + (\beta+1)\int_M |\Delta v|^{q-2}\Delta v \Delta v_t \mathrm{d}\mu \\ & = (\alpha+1)\int_M \lambda|u|^\alpha|v|^\beta v u_t \mathrm{d}\mu + (\beta+1)\int_M \lambda|u|^\alpha|v|^\beta u v_t \mathrm{d}\mu \\ & = \lambda\int_M R|u|^\alpha|v|^\beta uv \mathrm{d}\mu, \text{ using (14).} \end{aligned} \tag{15}$$

Finally, using (15) in (13), we obtain (10). □

Theorem 1. *Suppose $(M^n, g(t))$ is a solution of (6) on the smooth closed Riemannian manifold (M^n, g_0) and $\lambda(t)$ is the evolution of the first eigenvalue of the system (1), with the associated normalized eigenfunction $(u(t), v(t))$. If $R_{min}(0) > 0$ and $R_{ij} \geq \gamma R g_{ij}$ with $\frac{1}{k} \leq \gamma \leq \frac{1}{n}$, where $k = min\{p,q\}$, then the quantity*

$$\lambda(t)(R_{min}^{-1}(0) - \frac{2}{n}t)^{nk\gamma} \tag{16}$$

is monotone nondecreasing on $[0, T^{'})$*, where* $T^{'} = min\{T, \frac{n}{2}R_{min}(0)\}$*. Moreover,* $\lambda(t)$ *is differentiable almost everywhere on* $[0, T^{'})$*.*

Proof. Using the fact that $R_{ij} \geq \gamma R g_{ij}$, from (10) we have

$$\begin{aligned} \frac{d}{dt}\lambda(t, u(t), v(t))|_{t=t_0} \geq{} & 2\gamma(\alpha+1)\int_M |\Delta u|^p R \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M R|\Delta u|^p \mathrm{d}\mu \\ & + 2\gamma(\beta+1)\int_M |\Delta v|^q R \mathrm{d}\mu - \frac{\beta+1}{q}\int_M R|\Delta v|^q \mathrm{d}\mu \\ & + \lambda(t_0)\int_M R|u|^\alpha |v|^\beta uv \mathrm{d}\mu \\ \geq{} & (2p\gamma - 1)\frac{\alpha+1}{p}\int_M R|\Delta u|^p \mathrm{d}\mu \\ & + (2q\gamma - 1)\frac{\beta+1}{q}\int_M R|\Delta v|^q \mathrm{d}\mu \\ & + \lambda(t_0)\int_M R|u|^\alpha |v|^\beta uv \mathrm{d}\mu. \end{aligned} \tag{17}$$

From Lemma 2 and using the inequality $|Ric|^2 \geq \frac{1}{n}R^2$, we have

$$\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{n}R^2. \tag{18}$$

The solution to the ODE $\frac{d}{dt}\sigma(t) = \frac{2}{n}\sigma^2(t)$, $\sigma(0) = R_{min}(0)$ is

$$\sigma(t) = \frac{1}{(R_{min}(0))^{-1} - \frac{2}{n}t}. \tag{19}$$

Then, by using maximum principle we have

$$R(x,t) \geq \sigma(t) = \frac{1}{(R_{min}(0))^{-1} - \frac{2}{n}t}, \quad t \in [0, T^{'}) \tag{20}$$

where $T^{'} = min\{T, \frac{n}{2}R_{min}(0)\}$. Hence, from (17) in a sufficiently small neighborhood of t_0 we obtain

$$\frac{d}{dt}\lambda(t) \geq 2k\gamma\left(\frac{1}{(R_{min}(0))^{-1} - \frac{2}{n}t}\right)\lambda(t). \tag{21}$$

Taking integration of the above inequality on $[t_1, t_0]$, we obtain

$$\lambda(t_0) \geq \lambda(t_1)\exp\left\{2k\gamma\int_{t_1}^{t_0}\frac{dt}{(R_{min}(0))^{-1} - \frac{2}{n}t}\right\}, \tag{22}$$

i.e.,

$$\lambda(t_0)(R_{min}^{-1}(0) - \frac{2}{n}t_0)^{nk\gamma} \geq \lambda(t_1)(R_{min}^{-1}(0) - \frac{2}{n}t_1)^{nk\gamma}. \tag{23}$$

Hence, the quantity $\lambda(t)(R_{min}^{-1}(0) - \frac{2}{n}t)^{nk\gamma}$ is monotone nondecreasing on $[t_1, t_0]$. Since t_0 is arbitrary, $\lambda(t)(R_{min}^{-1}(0) - \frac{2}{n}t)^{nk\gamma}$ is monotonic nondecreasing on $[0, T^{'})$. Since $\lambda(t)$ is monotone and continuous on $[0, T^{'})$, then the classical Lebesgue theorem implies that $\lambda(t)$ is almost everywhere differentiable on $[0, T^{'})$. $\square$

Theorem 2. *Suppose* $(M^2, g(t))$ *is a solution of* (6) *on the closed surface* (M^2, g_0) *with non-negative scalar curvature. If* $\lambda(t)$ *is the evolution of the first eigenvalue of the system* (1)*, then* $\lambda(t)$ *is monotone nondecreasing.*

Proof. On a surface we have $R_{ij} = \frac{1}{2}Rg_{ij}$. So, from $\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2$ we obtain $\frac{\partial R}{\partial t} = \Delta R + R^2$. Using the maximum principle, one can demonstrate that the scalar curvature R remains non-negative under the Ricci flow. Now, from

$$\begin{aligned}\frac{d}{dt}\lambda(t, u(t), v(t))|_{t=t_0} &= \frac{\alpha+1}{p}(p-1)\int_M |\Delta u|^p R\mathrm{d}\mu \\ &+ \frac{\beta+1}{q}(q-1)\int_M |\Delta v|^q R\mathrm{d}\mu \\ &+ \lambda(t_0)\int_M R|u|^\alpha |v|^\beta uv\mathrm{d}\mu \\ &\geq 0,\end{aligned}$$

which shows that $\lambda(t)$ is monotone nondecreasing. □

Corollary 1. *Suppose* $(M^n, g(t))$ *is a solution of* (6) *on a closed homogeneous Riemannian manifold* (M^n, g_0)*. If* $\lambda(t)$ *is the evolution of the first eigenvalue of the system* (1)*, then*

$$\begin{aligned}\frac{d}{dt}\lambda(t, u(t), v(t))|_{t=t_0} &= 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u\mathrm{d}\mu \\ &+ 2(\beta+1)\int_M |\Delta v|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v\mathrm{d}\mu.\end{aligned}$$

Proof. Since scalar curvature on an evolving homogeneous manifold is constant, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t, u(t), v(t))|_{t=t_0} &= 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u\mathrm{d}\mu \\ &+ 2(\beta+1)\int_M |\Delta v|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v\mathrm{d}\mu \\ &- R\left(\frac{\alpha+1}{p}\int_M |\Delta u|^p\mathrm{d}\mu + \frac{\beta+1}{q}\int_M |\Delta v|^q\mathrm{d}\mu\right) \\ &+ \lambda(t_0)R\int_M |u|^\alpha|v|^\beta uv\mathrm{d}\mu,\end{aligned}$$

from which Corollary 1 follows. □

3.2. Variation in Eigenvalue along Normalized Ricci Flow

Normalized Ricci flow is given by the following equation:

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \frac{2}{n}rg_{ij}, \ \ g(0) = g_0, \tag{24}$$

where $r = \frac{\int_M R\mathrm{d}\mu}{\int_M \mathrm{d}\mu}$ is the average scalar curvature. Along the normalized Ricci flow (24), we have the following evolution equations:

$$(i)\frac{\partial}{\partial t}\mathrm{d}\mu = (r-R)\mathrm{d}\mu, \qquad (ii)\frac{\partial}{\partial t}\Delta u = 2R^{ij}\nabla_i\nabla_j u + \Delta u_t - \frac{2}{n}r\Delta u. \tag{25}$$

Proposition 4. *Suppose* $(M^n, g(t))$ *is a solution of* (24) *on closed Riemannian manifold* (M^n, g_0) *and* $\lambda(t)$ *is the evolution of the first eigenvalue of the system* (1)*. Then,*

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} &= 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M |\Delta u|^p R \mathrm{d}\mu \\ &+ 2(\beta+1)\int_M |\Delta u|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v \mathrm{d}\mu - \frac{\beta+1}{q}\int_M |\Delta v|^q R \mathrm{d}\mu \\ &- 2r\frac{\alpha+1}{n}\int_M |\Delta u|^p \mathrm{d}\mu - 2r\frac{\beta+1}{n}\int_M |\Delta v|^q \mathrm{d}\mu \\ &+ \lambda(t_0)\int_M |u|^\alpha |v|^\beta uvR\mathrm{d}\mu. \end{aligned} \tag{26}$$

Here, $(u(t), v(t))$ *is the associated normalized eigenfunction.*

Proof. Differentiating (7) with respect to the time t at $t = t_0$, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} &= \frac{\alpha+1}{p}\int_M \frac{\partial}{\partial t}(|\Delta u|^p)\mathrm{d}\mu + \frac{\beta+1}{q}\int_M \frac{\partial}{\partial t}(|\Delta v|^q)\mathrm{d}\mu \\ &+ \frac{\alpha+1}{p}\int_M |\Delta u|^p (r-R)\mathrm{d}\mu + \frac{\beta+1}{q}\int_M |\Delta v|^q (r-R)\mathrm{d}\mu. \end{aligned} \tag{27}$$

Now,

$$\begin{aligned}\frac{\partial}{\partial t}(|\Delta u|^p) &= \frac{p}{2}|\Delta u|^{p-2}\frac{\partial}{\partial t}(|\Delta u|^2) \\ &= p|\Delta u|^{p-2}\Delta u\{2R^{ij}\nabla_i\nabla_j u + \Delta u_t - \frac{2}{n}r\Delta u\}. \end{aligned} \tag{28}$$

Using (28) and (27) yields

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} &= \frac{\alpha+1}{p}\int_M p|\Delta u|^{p-2}\Delta u\{2R^{ij}\nabla_i\nabla_j u + \Delta u_t - \frac{2}{n}r\Delta u\}\mathrm{d}\mu \\ &+ \frac{\beta+1}{q}\int_M q|\Delta u|^{q-2}\Delta v\{2R^{ij}\nabla_i\nabla_j v + \Delta v_t - \frac{2}{n}r\Delta v\}\mathrm{d}\mu \\ &+ \frac{\alpha+1}{p}\int_M |\Delta u|^p (r-R)\mathrm{d}\mu + \frac{\beta+1}{q}\int_M |\Delta v|^q (r-R)\mathrm{d}\mu \\ &= 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M |\Delta u|^p R\mathrm{d}\mu \\ &+ 2(\beta+1)\int_M |\Delta u|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v \mathrm{d}\mu - \frac{\beta+1}{q}\int_M |\Delta v|^q R\mathrm{d}\mu \\ &+ (\alpha+1)\int_M |\Delta u|^{p-2}\Delta u\Delta u_t \mathrm{d}\mu + (\beta+1)\int_M |\Delta u|^{q-2}\Delta v\Delta v_t \mathrm{d}\mu \\ &- 2r\frac{\alpha+1}{n}\int_M |\Delta u|^p \mathrm{d}\mu - 2r\frac{\beta+1}{n}\int_M |\Delta v|^q \mathrm{d}\mu + r\lambda(t_0). \end{aligned} \tag{29}$$

Differentiating $\int_M |u|^\alpha |v|^\beta uv\mathrm{d}\mu = 1$ we obtain

$$(\alpha+1)\int_M |u|^\alpha |v|^\beta u_t v\mathrm{d}\mu + (\beta+1)\int_M |u|^\alpha |v|^\beta uv_t\mathrm{d}\mu = -\int_M |u|^\alpha |v|^\beta uv(r-R)\mathrm{d}\mu. \tag{30}$$

Thus,

$$\begin{aligned}(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u\Delta u_t\mathrm{d}\mu + (\beta+1)\int_M |\Delta u|^{q-2}\Delta v\Delta v_t\mathrm{d}\mu =& \lambda\int_M |u|^\alpha |v|^\beta uvR\mathrm{d}\mu \\ &- r\lambda. \end{aligned} \tag{31}$$

Substituting (31) into (29) we obtain the result. □

Theorem 3. *Suppose* $(M^2, g(t))$ *is a solution of* (24) *on closed Riemannian surface* (M^2, g_0)*. If* $\lambda(t)$ *is the evolution of the first eigenvalue of the system* (1)*, then*

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} =&(p-1)\frac{\alpha+1}{p}\int_M |\Delta u|^p R\mathrm{d}\mu + (q-1)\frac{\beta+1}{q}\int_M |\Delta v|^q R\mathrm{d}\mu \\ &- r(\alpha+1)\int_M |\Delta u|^p \mathrm{d}\mu - r(\beta+1)\int_M |\Delta v|^q \mathrm{d}\mu \\ &+ \lambda(t_0)\int_M |u|^\alpha |v|^\beta uvR\mathrm{d}\mu,\end{aligned} \tag{32}$$

where $(u(t), v(t))$ *is the associated normalized eigenfunction.*

Proof. On a closed surface, we have $R_{ij} = \frac{1}{2}Rg_{ij}$. Thus, from (26), we have

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} =& (\alpha+1)\int_M R|\Delta u|^p \mathrm{d}\mu - \frac{\alpha+1}{p}\int_M |\Delta u|^p R\mathrm{d}\mu \\ &+ (\beta+1)\int_M R|\Delta u|^q \mathrm{d}\mu - \frac{\beta+1}{q}\int_M |\Delta v|^q R\mathrm{d}\mu \\ &- r(\alpha+1)\int_M |\Delta u|^p \mathrm{d}\mu - r(\beta+1)\int_M |\Delta v|^q \mathrm{d}\mu \\ &+ \lambda(t_0)\int_M |u|^\alpha |v|^\beta uvR\mathrm{d}\mu.\end{aligned} \tag{33}$$

Hence, the result follows. □

Corollary 2. *Suppose* $(M^n, g(t))$ *is a solution of* (24) *on a closed homogeneous Riemannian manifold* (M^n, g_0)*. If* $\lambda(t)$ *is the evolution of the first eigenvalue of the system* (1)*, then*

$$\begin{aligned}\frac{d}{dt}\lambda(t,u(t),v(t))|_{t=t_0} =& 2(\alpha+1)\int_M |\Delta u|^{p-2}\Delta u R^{ij}\nabla_i\nabla_j u\mathrm{d}\mu \\ &+ 2(\beta+1)\int_M |\Delta u|^{q-2}\Delta v R^{ij}\nabla_i\nabla_j v\mathrm{d}\mu \\ &- 2r\frac{\alpha+1}{n}\int_M |\Delta u|^p \mathrm{d}\mu - 2r\frac{\beta+1}{n}\int_M |\Delta v|^q \mathrm{d}\mu.\end{aligned}$$

Proof. This result has the same proof of Corollary 1. □

4. Conclusions and Future Expectations

Harmonic functions plays a significant role in Dirichlet boundary value and Neumann boundary value problems. As we know, a smooth function $u : M \to \mathbb{R}$ is called a harmonic function if $\Delta u = 0$. The function u is called biharmonic if $\Delta^2 u = 0$. Here, Δ^2 is known as the biharmonic operator. The biharmonic function has application to the continuum mechanics and elasticity theory. It is known that every harmonic function is biharmonic, but the converse is not true. As a generalization, the function u is called p-biharmonic if $\Delta_p^2 u = 0$ for $p \in (1, +\infty)$, where Δ_p^2 is known as the p-biharmonic operator (an elliptic operator of fourth order), defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$. For $p = 2$, the p-biharmonic operator reduces to a harmonic operator. Based on these definitions and motivations, in this paper we studied the variation formula of the first eigenvalue of the (p, q)-biharmonic system on a closed Riemannian manifold. We also obtained some monotonic quantities. In future work, we want to perform interdisciplinary research addressing soliton theory, singularity theory, submanifold theory, etc., to find more new results. We will take advantage of those theories and results presented in [13–25] to adapt and improve the approaches to develop flexible methods to study the eigenvalues of geometric operators. To study the mechanical vibrations of plates, the eigenvalue problems of biharmonic systems play an important role. Therefore, in the future research we also want to explore the applications in engineering, nuclear physics, signal processing, etc.

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