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Unified Sufficient Conditions for Predefined-Time Stability of Non-Linear Systems and Its Standard Controller Design

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Abstract: This paper presents a unified Lyapunov-based predefined-time stability theorem that includes three sufficient conditions. The standard theoretical analysis method for achieving predefined-time stability of non-linear systems using this theorem is provided within the framework of Lyapunov theory. The developed Lyapunov-based theorem facilitates the establishment of equivalence between the existing Lyapunov theorems concerning predefined-time stability. Furthermore, when the presented sufficient conditions are relaxed, the predefined-time stability conclusion for non-linear systems degenerates into a finite-time one. Consequently, a standard non-singular sliding mode control framework based on the unified Lyapunov-based theorem is developed for a Lagrangian system to ensure its predefined-time stability. Exemplary numerical simulation results are subsequently given, in order to illustrate the convergence behavior of the system states and confirm that the controlled systems are predefined-time stable.

Keywords: predefined-time stability; non-linear control system; sliding mode control; Lyapunov theory



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1. Introduction

The convergence rate is an important index for evaluating the control performance of a controlled system. The finite-time control method was introduced to attain rapid stability [1], thereby addressing limitations inherent in asymptotically stable control systems characterized by infinite convergence time. The convergence time is contingent upon the system's initial states, resulting in a variable settling time corresponding to different initial states that becomes indeterminate. The fixed-time control method was defined to make the upper bound on the convergence time less conservative [2,3], and severs the dependence of settling time on the system's initial states. However, it involves a complex function with multiple control gains to characterize the upper bound of the convergence time, making it challenging to arbitrarily select a convergence time through the tuning of multiple control gains. To this end, the predefined-time control technique, with the settling time as an explicit parameter that can be determined in advance, was discussed in [4–8]. This approach has greater flexibility in determining the settling time than finite- and fixed-time control methodologies, thus facilitating the design of observers and controllers that are suitable for addressing challenges necessitating adherence to rigorous time constraints.

Lyapunov theory—an effective instrument for analyzing the stability of control systems—is often combined with sliding mode control, backstepping control, and adding power integrator techniques to design controllers to ensure asymptotic [9], finite-time [10–12], fixed-time [13–15], and predefined-time [16,17] stability of the controlled system. In particular, the sliding mode control technique is usually used to design a controller to guarantee the controlled system's finite-time stability [17–22]. As detailed in [4,23,24], some paradigms

of the predefined-time controller were formulated to achieve the control requirements of second-order non-linear systems featuring uncertainties, but challenges related to the singularity of sliding mode control were encountered. To overcome this drawback, alternative non-singular predefined-time sliding mode controllers were presented in [17,20,25]. The aforementioned predefined-time controllers, designed using different Lyapunov-based predefined-time stability theorems, can be applied to many systems and exhibit predefined-time stability properties. In fact, through analyzing these theorems, one can derive a unified Lyapunov theorem that covers the existing Lyapunov theorems presented in [6,19,23,26], equally ensuring the predefined-time stability of non-linear systems.

On the basis of Lyapunov theory, this study develops a unified Lyapunov-based theorem guaranteeing the predefined-time/finite-time stability of non-linear systems. The main contributions are stated as follows:

- (1) A unified Lyapunov theorem with three sufficient conditions is proposed, which guarantees that non-linear systems achieve predefined-time stability. This differs from that reported in [6], which stated that a strictly increasing \mathcal{K}_1 regulator function is required for the Lyapunov-based predefined-time stability theorem to hold true, thus restricting the Lyapunov-based predefined-time theorem's selection. The results presented in this paper relax this constraint. The new Lyapunov-based predefined-time stability theorem allows for the use of an arbitrary, strictly monotonically bounded increasing or decreasing regulator function. Moreover, it serves to unify the Lyapunov-based predefined-time stability theorems for non-linear systems previously published in the literature [6,19,25–27].
- (2) A unified finite-time stability solution for non-linear systems using Lyapunov theory is derived. Despite the widespread application of the Lyapunov-based finite-time stability theorem, this study's results not only uncover additional potential Lyapunov-based finite-time stability theorems through the selection of different strictly monotonically unbounded increasing or decreasing regulator functions but also cover the existing Lyapunov-based finite-time stability theorems in [28–30].
- (3) Using the sliding mode control technique and the proposed unified Lyapunov-based predefined-time stability theorem, a class of non-singular predefined-time sliding mode control frameworks is developed for a second-order Lagrangian system, ensuring its predefined-time stability. Simulation examples further substantiate the effectiveness of the aforementioned control method, and the simulation results provide a comprehensive exposition of the proposed controller's behavior, including its control accuracy and settling time.

The subsequent sections of this work are organized as follows: The preliminaries and motivation are given in Section 2. Section 3 presents the unified predefined-/finite-time stability conclusions. A standard predefined-time controller design method is developed in Section 4. The numerical simulation examples carried out to validate the effectiveness of the proposed control method are detailed in Section 5. Finally, Section 6 concludes this paper.

2. Preliminaries and Motivation

2.1. Notation

For the convenience of reading, the following notation is used: \mathfrak{R} is a set of real numbers. Let \mathfrak{R}^m be the m -dimensional Euclidean space. For $\chi \in \mathfrak{R}^n$, χ^\top denotes its transpose, and $\dot{\chi} = \frac{d\chi}{dt}$ denotes the time derivative of χ . For a scalar function $h(\chi) : \mathfrak{R}^n \rightarrow \mathfrak{R}$, $e^{h(\chi)}$ represents a standard exponential function with a natural constant e as the base, and $|h(\chi)|$ denotes the absolute value of $h(\chi)$. A function $P(\chi) : [0, \infty) \rightarrow [0, a)$ is said to be a class \mathcal{K}_a function if it is strictly increasing with $P(0) = 0$.

2.2. Definitions

Consider the following non-linear system:

$$\dot{x} = f(x; u), \quad (1)$$

where $x \in \mathfrak{R}^n$ is the system state, $u \in \mathfrak{R}^m$ stands for the control input, and x_0 is the system's initial state. The origin $x = \mathbf{0}$ is the unique equilibrium of the system (1). The smooth function $f : \mathfrak{R}^n \times \mathfrak{R}^m \rightarrow \mathfrak{R}^n$ such that the solutions $\Psi(t, x_0)$ of system (1) exist and are unique in the sense of Filippov.

Definition 1 ([6]). *The origin of system (1) is said to be Lyapunov stable if, for any initial state $x_0 \in \mathfrak{R}^n$, the solution $\Psi(t, x_0)$ is well defined for all $t \geq 0$. Additionally, for any arbitrarily chosen $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that, for any $x_0 \in \mathfrak{R}^n$ satisfying $x_0 \in B_\delta(\mathbf{0})$, it follows that $\Psi(t, x_0) \in B_\epsilon(\mathbf{0})$ for all $t \geq 0$.*

Definition 2 ([3]). *The origin of system (1) is regarded as having asymptotic stability when it exhibits Lyapunov stability and satisfies the condition that $\Psi(t, x_0) \rightarrow 0$ as $t \rightarrow \infty$ for any initial state vector $x_0 \in \mathfrak{R}^n$.*

Definition 3 ([3]). *The origin of system (1) is deemed finite-time stable if it satisfies Lyapunov stability and, for any $x_0 \in \mathfrak{R}^n$, there exists a settling time function $T(x_0)$ such that $\Psi(t, x_0) = \mathbf{0}$ for all $t \geq T(x_0)$.*

Definition 4 ([3]). *The origin of system (1) is considered to be fixed-time stable if it satisfies finite-time stability and if its settling time, denoted by $T(x_0)$, is constrained by a bound; that is, there exists T_{\max} such that $\sup_{x_0 \in \mathfrak{R}^n} T(x_0) \leq T_{\max} < \infty$.*

Definition 5 ([6]). *The origin of system (1) is designated as predefined-time stable if it satisfies fixed-time stability and, for any predefined time constant $T_c > 0$, the settling time of system (1) adheres to the condition $\sup_{x_0 \in \mathfrak{R}^n} T(x_0) \leq T_c$.*

2.3. Motivating Example

Consider the common finite-time stable first-order scalar system [31].

$$\dot{x} = -kx^\eta, \quad (2)$$

with $k > 0$ and $0 < \eta < 1$, where x represents the system state. Let x_0 be the initial value of the system state. Its state trajectory is finite-time stable by Lyapunov analysis with a positive function $V = \frac{1}{2}x^2$ selected. The upper bound of the convergence time can be determined as $t \leq \frac{1}{k(1-\eta)2^{\frac{\eta-1}{2}}} |V_0|^{\frac{1-\eta}{2}}$ with $V_0 = \frac{1}{2}x_0^2$. When $x_0 \rightarrow +\infty$, one obtains $V_0 \rightarrow +\infty$. As the initial value increases, the convergence time will also synchronously increase. Hence, we will give a solution that the system's settling time is independent of the initial condition and can be assigned arbitrarily. Hence, for the dynamics system (2), we can rewrite its right-hand control input to make it a predefined-time stable system as follows:

$$\dot{x} = -\frac{1}{2pT_c} e^{Vp} V^{-p} x, \quad (3)$$

with $0 < p < 1$ and $T_c > 0$ being the predefined time constant. Taking the time derivative of V yields

$$\frac{dV}{dt} = -\frac{1}{pT_c} e^{Vp} V^{1-p} < 0, \quad \forall x \neq 0. \quad (4)$$

Intuitively, the scalar system (3) is asymptotically stable due to the fact that $e^{V^p} > 0$ always holds. Simultaneously, we can compute its convergence time. Rewriting (4) yields

$$\frac{d\psi(V)}{dt} = -\frac{1}{T_c}, \tag{5}$$

where $\psi(V) = 2 - e^{-V^p}$ is an increasing function with $\psi(V) \in [1, 2)$. Therefore, $\psi(V)$ will decrease to the minimum $\psi_T = 1$ from any initial value $\psi(V_0) < 2$. Integrating (5) yields

$$\int_{\psi(V_0)}^{\psi_T} 1 d\psi = -\int_0^T \frac{1}{T_c} dt. \tag{6}$$

We simplify (6) to obtain $T = (\psi(V_0) - \psi_T)T_c \leq T_c$. From (5) and (6), it can be proved that $T \leq T_c$ is valid for any $\psi(V_0)$ decreasing to the minimum of $\psi(V)$. The Lyapunov candidate V also converges to zero simultaneously, and the selected V is radially unbounded. For any initial system state x_0 , it can converge to the equilibrium when $T > T_c$.

3. Unified Predefined-/Finite-Time Stability Theorem

This section explores the sufficient conditions for ensuring that non-linear systems exhibit predefined-/finite-time stability within the framework of Lyapunov theory. We endeavor to establish a unified Lyapunov-based predefined-/finite-time stability theorem that not only can yield more potential Lyapunov-based predefined-/finite-time stability theorems but also covers the existing Lyapunov predefined-/finite-time stability theorems through the selection of different regulator functions.

Theorem 1. For system (1), if there exists a regulator function $\psi(V)$, with V being a positive, radially unbounded function, and the following three sufficient conditions are satisfied:

- (i) $\forall x \in \mathfrak{R}^n, \psi(V) \in [a, b), \psi(0) = a$ with $a \in \mathfrak{R}$ and $b \in \mathfrak{R}$;
- (ii) $\forall x \in \mathfrak{R}^n, \frac{d\psi}{d(V^p)} > 0$ with $0 < p < 1$;
- (iii) $\forall x \neq 0, \frac{dV}{dt} \leq -\frac{b-a}{\frac{d\psi}{d(V^p)}} \frac{V^{1-p}}{pT_c}$ with $T_c > 0$,

then system (1) is predefined-time stable, and the upper bound of the settling time is T_c .

Proof. Remember the system (1) and the candidate Lyapunov chosen as $V = \frac{1}{2}x^T x$. If the planned control input can make the Lyapunov function V meet sufficient condition (iii) in Theorem 1, then it is easy to observe that the system’s origin is Lyapunov asymptotic stable. Next, taking the time derivative of ψ and invoking sufficient condition (iii) in Theorem 1 yields

$$\frac{d\psi}{dt} = \frac{d\psi}{dV} \frac{dV}{dt} = pV^{p-1} \frac{d\psi}{d(V^p)} \frac{dV}{dt} \leq -\frac{b-a}{T_c}. \tag{7}$$

From (7), the function $\psi(V)$ will stabilize at its minimum $\psi_T = \psi(0) = a$ from any initial value $\psi_0 = \psi(V_0) < b$. Therefore, the selected V also decreases to zero simultaneously. Then, we calculate the convergence time of $\psi(V)$ from the initial state ψ_0 to the minimum ψ_T . Integrating both sides $d\psi \leq -\frac{b-a}{T_c} dt$ with respect to time, we have

$$\int_{\psi_0}^{\psi_T} d\psi \leq -\int_0^T \frac{b-a}{T_c} dt. \tag{8}$$

As a consequence, the following inequality holds:

$$T \leq \frac{(\psi_0 - \psi_T)T_c}{b-a} = \frac{(\psi_0 - a)T_c}{b-a} \leq T_c. \tag{9}$$

Hence, $\psi(V)$ decreases from an arbitrary initial value ψ_0 to the minimum ψ_T , whose convergence time satisfies $T \leq T_c$. Through the introduction of the function $\psi(V)$, as it

undergoes a decrease from its initial value to the minimum within the predefined time T_c , its independent variable V simultaneously converges to the origin. Consequently, the state of system (1), starting from any arbitrary initial state x_0 , exhibits convergence toward the equilibrium state $x \equiv \mathbf{0}$ when $T \geq T_c$. \square

Through the above analysis, a unified predefined-time stability theorem using the Lyapunov theory has been presented. The choice of function $\psi(V)$ is critical to obtaining the Lyapunov theorem and predefined-time stability. Moreover, the existing Lyapunov predefined-time stability theorems documented in the literature [19,23,26] can be regarded as special cases of Theorem 1. Several examples are given in the following:

Example 1. A specified regulator function $\psi(V) = b - e^{-\alpha V^p}$ is selected, with $b \in \mathfrak{R}$, $\alpha > 0$, and $0 < p < 1$. The fact that $\psi(V) \in [b - 1, b)$ always holds implies that $b - a = 1$. Differentiating $\psi(V)$ with respect to V^p , one has $d\psi(V)/dV^p = \alpha e^{-\alpha V^p}$. Thus, sufficient condition (iii) in Theorem 1 becomes

$$\frac{dV}{dt} \leq -\frac{b-a}{\alpha p T_c} e^{\alpha V^p} V^{1-p}, \forall x \neq \mathbf{0}. \tag{10}$$

Using Theorem 1, the system (1) is predefined-time stable with convergence time such that $T < T_c$. When selecting $\alpha = 1$, formula (10) reduces to $\frac{dV}{dt} \leq -\frac{1}{p T_c} e^{V^p} V^{1-p}, \forall x \neq \mathbf{0}$. Hence, this specific case covers the results in [23].

Example 2. A particular selection of the regulator function $\psi(V) = \arcsin(\tanh(V^p))$ is given, with $0 < p < 1$. Using simple mathematical operations, one has $\psi(V) \in [0, \frac{\pi}{2})$. Therefore, $b - a = \frac{\pi}{2}$. Differentiating $\psi(V)$ with respect to V^p , it can be deduced that $d\psi(V)/dV^p = \sqrt{1 - \tanh^2(V^p)}$. The sufficient condition (iii) in Theorem 1 is given as follows:

$$\frac{dV}{dt} \leq -\frac{b-a}{p T_c} \cosh(V^p) V^{1-p}, \forall x \neq \mathbf{0}. \tag{11}$$

On the basis of Theorem 1, system (1) is stable within predefined time T_c . Further, Formula (11) can be rewritten as $\frac{dV}{dt} \leq -\frac{\pi}{2 p T_c} \cosh(V^p) V^{1-p}, \forall x \neq \mathbf{0}$. This is identical to the result provided in [26].

Example 3. Choosing a regulator function $\psi(V) = \arctan(\sqrt{\frac{\beta}{\alpha}} V^p)$ with $\alpha > 0$, $\beta > 0$, and $0 < p < 1$, one can conclude that $\psi(V) \in [0, \frac{\pi}{2})$; that is, $b - a = \frac{\pi}{2}$. Differentiating $\psi(V)$ with respect to V^p , it can be derived as $d\psi(V)/dV^p = \sqrt{\alpha\beta}/(\alpha + \beta V^{2p})$. Thus, the sufficient condition (iii) in Theorem 1 becomes

$$\frac{dV}{dt} \leq -\frac{b-a}{p \sqrt{\alpha\beta} T_c} (\alpha V^{1-p} + \beta V^{1+p}), \forall x \neq \mathbf{0}. \tag{12}$$

Using Theorem 1, the state of system (1) converges to the origin when $T \geq T_c$. If $\alpha = 1$ and $\beta = 1$ are selected, the inequality (12) reduces to $\dot{V} \leq -\frac{\pi}{2 p T_c} (V^{1-p} + V^{1+p})$, which is equivalent to Theorem 1 of [19] with the special selection.

Example 4. To achieve predefined-time stability of system (1), a generalized Lyapunov-like theorem has been provided in [6]. As the inequality $\dot{\psi}(V) \leq -\frac{1}{(1-p)T_c} \psi^p(V)$ needs to be satisfied and the derived settling time function $T = T_c(\psi(V_0))^{1-p} \leq T_c$ should be met, it requires a strict constraint that the function $\psi(V) \in [0, 1)$ is a \mathcal{K}_1 function. In contrast, the theorem reported in this study breaks this constraint. Only an arbitrary increasing function $\psi(V) \in [a, b)$ is needed, increasing the flexibility in selecting $\psi(V)$. As such, the unified Lyapunov-based predefined-time stability Theorem 1 was derived, which can ensure the predefined-time stability of the controlled system. Due to the fact that the interval $[0, 1)$ can be included in the interval $[a, b)$, the results in [6] are covered by Theorem 1.

Therefore, the aforementioned Lyapunov-based predefined-time stability examples are covered by Theorem 1 with a specific regulator function $\psi(V)$.

Corollary 1. For system (1), if there is a regulator function $\psi(V)$, with V being a positive and radically unbounded function, and the following three sufficient conditions are satisfied:

- (i) $\forall \mathbf{x} \in \mathfrak{R}^n, \psi(V) \in (a, b], \psi(0) = b$ with $a \in \mathfrak{R}$ and $b \in \mathfrak{R}$;
- (ii) $\forall \mathbf{x} \in \mathfrak{R}^n, \frac{d\psi}{d(V^p)} < 0$ with $0 < p < 1$;
- (iii) $\forall \mathbf{x} \neq \mathbf{0}, \frac{dV}{dt} \leq \frac{b-a}{a(V^p)} \frac{V^{1-p}}{pT_c}$ with $T_c > 0$,

then system (1) is predefined-time stable, and the upper bound of the settling time is T_c .

Proof. Consider the system (1) and the regulator function $\psi(V)$ with $V = \frac{1}{2}\mathbf{x}^\top \mathbf{x}$. Develop a control input to make the Lyapunov function V satisfy the sufficient condition (iii) in Corollary 1. The straightforward derivation of the result regarding the system's origin is asymptotically stable.

Then, taking the time derivative of $\psi(V)$ and invoking sufficient condition (iii) in Corollary 1 yields $\frac{d\psi}{dt} \geq \frac{b-a}{T_c}$, which means that the function $\psi(V)$ will stabilize at its maximum $\psi_T = \psi(0) = b$ from any initial value $\psi_0 = \psi(V_0) > a$. Therefore, the selected V also decreases to zero simultaneously. Then, we calculate the convergence time of $\psi(V)$ from the initial state ψ_0 to the maximum ψ_T . Integrating both sides $d\psi \geq \frac{b-a}{T_c} dt$, one obtains $\int_{\psi_0}^{\psi_T} d\psi \geq \int_0^T \frac{b-a}{T_c} dt$. As a consequence, the following holds:

$$T \leq -\frac{(\psi_0 - \psi_T)T_c}{b - a} = \frac{(b - \psi_0)T_c}{b - a} \leq T_c. \quad (13)$$

Hence, $\psi(V)$ from arbitrary initial value ψ_0 to the maximum ψ_T has convergence time such that $T \leq T_c$. When the function $\psi(V)$ increases from the initial value to the maximum value within the predefined time T_c , its independent variable V also converges synchronously to the origin. Therefore, the system state in (1) from the arbitrary initial state \mathbf{x}_0 converges to the equilibrium $\mathbf{x} \equiv \mathbf{0}$ when $T > T_c$. \square

Summarizing the analysis of Theorem 1 and Corollary 1, the predefined-time stability of system (1) can be guaranteed. On this basis, we can obtain a series of predefined-time stable dynamic systems. We give the standard predefined-time controller design process as follows:

- S1. Choose a strictly monotonically increasing function $\psi_i(\cdot)$ or decreasing function $\psi_d(\cdot)$;
- S2. Choose a positive Lyapunov candidate $V = \frac{1}{2}\mathbf{x}^\top \mathbf{x}$ to obtain the functions $\psi_i(V^p)$ and $\psi_d(V^p)$, with $0 < p < 1$. The two functions satisfy $\psi_i(\cdot) \in [a, b]$ with $\psi_i(0) = a$ and $\psi_d(\cdot) \in (a, b]$, where $\psi_d(0) = b$;
- S3. Take the derivatives of $\psi_i(V^p)$ and $\psi_d(V^p)$ with respect to V^p ;
- S4. Construct a Lyapunov inequality form of \dot{V} to meet the following inequalities: $\dot{\psi}_i(V^p) < -1/T_c$ and $\dot{\psi}_d(V^p) > 1/T_c$;
- S5. Design a control input u to meet the Lyapunov form of \dot{V} in step S4 and guarantee the predefined-time stability of the system (1).

Summarizing the above analysis, let the Lyapunov function V be the independent variable of the strictly monotonically increasing/decreasing bounded function $\psi_i(V) \in [a, b]$ or $\psi_d(V) \in (a, b]$. Utilizing the characteristics of the monotonic regulator function $\psi_i(V)$ or $\psi_d(V)$ forces V to decay to zero with the regulator function decreasing/increasing to its minimum/maximum. Therefore, we establish an equivalent relationship between the system's convergence and the decreasing/increasing characteristics of the regulator function, further reflecting the system's settling time. The selection of the regulator functions $\psi_i(\cdot)$ and $\psi_d(\cdot)$ is key to achieving predefined-time stability. Some examples of regulator functions are listed in Table 1.

Table 1. Candidate examples for Theorem 1 and Corollary 1.

Regulator Functions	Predefined-Time Stability Theorem	Parameters
$\psi_i = \frac{qV^p}{V^p+k}$,	$\dot{V} \leq -\frac{1}{kpT_c}(V^p+k)^2V^{1-p}$	$0 < p < 1, q > 0, k > 0, T_c > 0$
$\psi_i = n \tanh(mV^p)$,	$\dot{V} \leq -\frac{1}{4mpT_c}(e^{mV^p} + e^{-mV^p})^2V^{1-p}$	$0 < p < 1, n > 0, m > 0, T_c > 0$
$\psi_i = m \int_{-\frac{\ln n}{m}}^{z_1} e^{-mx} dx$,	$\dot{V} \leq -\frac{1}{mpT_c}(V^p+1)^{m+1}V^{1-p}$	$z_1 = \ln(V^p+1), 0 < p < 1, n > 1, m > 0, T_c > 0$
$\psi_i = (\frac{V}{\sqrt{V^2+\epsilon}})^p$,	$\dot{V} \leq -\frac{1}{\epsilon p T_c}(V^2+\epsilon)^{1+\frac{p}{2}}V^{1-p}$	$0 < p < 1, \epsilon > 0, T_c > 0$
$\psi_i = q - e^{-\alpha V^p}$,	$\dot{V} \leq -\frac{1}{\alpha p T_c}e^{\alpha V^p}V^{1-p}$	$0 < p < 1, \alpha > 0, q > 0, T_c > 0$
$\psi_i = \arcsin(\tanh(V^p))$,	$\dot{V} \leq -\frac{\pi}{2pT_c} \cosh(V^p)V^{1-p}$	$0 < p < 1, T_c > 0$
$\psi_i = \arctan(\sqrt{\frac{\beta}{\alpha}}V^p)$,	$\dot{V} \leq -\frac{\pi}{2p\sqrt{\alpha\beta}T_c}(\alpha V^{1-p} + \beta V^{1+p})$	$0 < p < 1, \alpha > 0, \beta > 0, T_c > 0$
$\psi_d = q + e^{-\alpha V^p}$,	$\dot{V} \leq -\frac{1}{\alpha p T_c}e^{\alpha V^p}V^{1-p}$	$0 < p < 1, \alpha > 0, q > 0, T_c > 0$
$\psi_d = \frac{n}{m+e^{\alpha V^p}}$,	$\dot{V} \leq -\frac{1}{\alpha p(m+1)T_c}(m+e^{\alpha V^p})^2e^{-\alpha V^p}V^{1-p}$	$0 < p < 1, \alpha > 0, m > 0, n > 0, T_c > 0$
$\psi_d = \frac{1}{V^p+k}$,	$\dot{V} \leq -\frac{1}{kpT_c}(V^p+k)^2V^{1-p}$	$0 < p < 1, k > 0, T_c > 0$
...

Moreover, if the regulator function $\psi(V)$ in Theorem 1 does not meet the sufficient condition (i), and $\psi(V) \in [a, +\infty)$ is a strictly monotonically increasing unbounded function, the conclusion of Theorem 1 degenerates into a finite-time one. The following unified finite-time stability corollaries based on Theorem 1 are presented.

Corollary 2. For the system (1), if there is a regulator function $\psi(V)$, with V being a positive and radically unbounded function, and the following three sufficient conditions are satisfied:

- (i) $\forall x \in \mathfrak{R}^n, \psi(V) \in [a, +\infty), \psi(0) = a$ with $a \in \mathfrak{R}$;
- (ii) $\forall x \in \mathfrak{R}^n, \frac{d\psi}{d(V^p)} > 0$ with $0 < p < 1$;
- (iii) $\forall x \neq 0, \frac{dV}{dt} \leq -\frac{1}{\frac{d\psi}{d(V^p)} p T_c} V^{1-p}$ with $T_c > 0$,

then system (1) is finite-time stable, and the upper bound of the settling time is $(\psi(V_0) - a)T_c$.

Proof. The upper bound of the settling time can also be computed using a similar method as in Theorem 1. Now, differentiating $\psi(V)$ and invoking condition (iii) in Corollary 2, one obtains $\frac{d\psi(V)}{dt} \leq -\frac{1}{T_c}$. Therefore, $\psi(V)$ will decrease to the minimum $\psi_T = a$ from the initial value $\psi(V_0)$. Integrating it on both sides yields $\int_{\psi(V_0)}^{\psi_T} 1d\psi \leq -\int_0^T \frac{1}{T_c} dt$. Then, we have $T \leq (\psi(V_0) - a)T_c$. Thus, it is proved that any $\psi(V_0)$ decreases to the minimum of $\psi(V)$ when $T > (\psi(V_0) - a)T_c$. The Lyapunov candidate V also converges to zero simultaneously. The selected V is radially unbounded. For any initial system state x_0 , it can converge to the equilibrium when $T > (\psi(V_0) - a)T_c$. Therefore, we can further summarize the following Lyapunov-based finite-time stability corollary on the basis of Corollary 1. \square

Corollary 3. For system (1), if there is a regulator function $\psi(V)$, with V being a positive and radically unbounded function, and the following three sufficient conditions are satisfied:

- (i) $\forall \mathbf{x} \in \mathfrak{R}^n, \psi(V) \in (-\infty, a], \psi(0) = a$ with $a \in \mathfrak{R}$;
- (ii) $\forall \mathbf{x} \in \mathfrak{R}^n, \frac{d\psi}{d(V^p)} < 0$ with $0 < p < 1$;
- (iii) $\forall \mathbf{x} \neq \mathbf{0}, \frac{dV}{dt} \leq \frac{1}{\frac{d\psi}{d(V^p)}} \frac{V^{1-p}}{pT_c}$ with $T_c > 0$,

then system (1) is finite-time stable, and the upper bound of the settling time is $(a - \psi(V_0))T_c$.

Proof. The analysis process is similar to Corollary 1 and, thus, is omitted here. \square

Selection of the regulator function $\psi(V)$ is important in deriving the Lyapunov-based finite-time stability theorem and ensuring the finite-time stability of the controlled system. Moreover, the existing Lyapunov-based finite-time stability theorems documented in the literature [28–30] can be regarded as special cases of Corollary 2. Several examples are given in the following:

Example 5. A specified regulator function $\psi(V) = kV^p$ is selected with $k > 0$ and $0 < p < 1$, such that $\psi(V) \in [0, +\infty)$ always holds. Differentiating $\psi(V)$ with respect to V^p , one has $d\psi(V)/dV^p = k$. Thus, sufficient condition (iii) in Corollary 2 becomes

$$\frac{dV}{dt} \leq -\frac{1}{kpT_c} V^{1-p}, \forall \mathbf{x} \neq \mathbf{0}. \quad (14)$$

Using Corollary 2, system (1) is finite-time stable when $T > (kV_0^p)T_c$, where $T_c > 0$ is a time constant. When selecting $k = 1$, formula (14) reduces to $\frac{dV}{dt} \leq -\frac{1}{pT_c} V^{1-p}, \forall \mathbf{x} \neq \mathbf{0}$. Hence, this special case covers the results in [28,29].

Example 6. A special selection of $\psi(V) = \ln(V^p + 1)$ is chosen, with $0 < p < 1$. Using simple mathematical operations, one has $\psi(V) \in [0, +\infty)$. Differentiating $\psi(V)$ with respect to V^p , it can be derived as $d\psi(V)/dV^p = 1/(V^p + 1)$. The sufficient condition (iii) in Corollary 2 is given as follows:

$$\frac{dV}{dt} \leq -\frac{1}{pT_c} V^{1-p} - \frac{1}{pT_c} V, \forall \mathbf{x} \neq \mathbf{0}. \quad (15)$$

On the basis of Corollary 2, the system (1) is stable within finite time $\ln(V_0^p + 1)T_c$, where $T_c > 0$ is a time constant. This is identical to the result of [30].

Summarizing the above analysis for Corollaries 2 and 3, the finite-time stability of system (1) can be guaranteed. On this basis, we can obtain a series of finite-time stable dynamic systems. Let the candidate Lyapunov function V be the independent variable of the strictly monotonically increasing/decreasing unbounded function $\psi_i(V) \in [a, \infty)$ or $\psi_d(V) \in (-\infty, b]$. Utilizing the characteristics of the monotonic regulator function $\psi_i(V)$ or $\psi_d(V)$ forces V to decay to zero with the regulator function decreasing/increasing to its minimum/maximum. Therefore, we establish an equivalent relationship between the system's convergence and the decreasing/increasing characteristics of the regulator function, which further reflects the system's settling time. Therefore, the selection of a decreasing/increasing regulator function is important in the derivation of Corollaries 2 and 3. Some possible regulator functions are listed in Table 2.

Table 2. Candidate examples for Corollaries 2 and 3.

Regulator Functions	Finite-Time Stability Condition	Parameters	Upper Bound of Settling Time
$\psi_i = \ln(mV^p + 1)$	$\dot{V} \leq -\frac{1}{mpT_c}(mV + V^{1-p})$	$0 < p < 1, m > 0, T_c > 0$	$T \leq \ln(mV_0^p + 1)T_c$
$\psi_i = \beta V^p$	$\dot{V} \leq -\frac{1}{\beta p T_c} V^{1-p}$	$0 < p < 1, \beta > 0, T_c > 0$	$T \leq \beta V_0^p T_c$
$\psi_i = e^{V^p}$	$\dot{V} \leq -\frac{1}{pT_c} e^{-V^p} V^{1-p}$	$0 < p < 1, T_c > 0$	$T \leq (e^{V_0^p} - 1)T_c$
$\psi_i = \sqrt{V^p + 1}$	$\dot{V} \leq -\frac{2}{pT_c} \sqrt{V^p + 1} V^{1-p}$	$0 < p < 1, T_c > 0$	$T \leq (\sqrt{V_0^p + 1} - 1)T_c$
$\psi_d = -\alpha V^p$	$\dot{V} \leq -\frac{1}{\alpha p T_c} V^{1-p}$	$0 < p < 1, \alpha > 0, T_c > 0$	$T \leq \alpha V_0^p T_c$
...

4. Standard Predefined-Time Controller Design

Due to the fact that many actual mechanical systems, including robot manipulator systems [32], unmanned aerial vehicles [33,34], and spacecraft attitude control systems [35,36], are typically described by the Lagrangian dynamic equations of motion, the study objective of this section is to develop a standard predefined-time controller design method for the Lagrangian system considering external disturbances. The associated mathematical model is given as follows:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + F_u = u, \tag{16}$$

where $q \in \mathfrak{R}^n, \dot{q} \in \mathfrak{R}^n; \ddot{q} \in \mathfrak{R}^n$ denote the generalized position, velocity, and acceleration vectors, respectively; $M(q) \in \mathfrak{R}^{n \times n}$ is a positive definite moment of inertia matrix; $C(q, \dot{q}) \in \mathfrak{R}^{n \times n}$ is the centripetal Coriolis matrix; $G(q) \in \mathfrak{R}^n$ is the gravity vector; $F_u \in \mathfrak{R}^n$ represents the uncertainty vector; and $u \in \mathfrak{R}^n$ denotes the control input. For the convenience of controller design, let $x_1 = q$ and $x_2 = \dot{q}$. Therefore, the Lagrangian system (16) with $n = 1$ can be rewritten as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + g(x_1, x_2)u + w' \end{cases} \tag{17}$$

where $x = [x_1 \ x_2]^T \in \mathfrak{R}^2$ is the available state vector; $g(x_1, x_2) = M^{-1}(x_1)$ and $f(x_1, x_2) = M^{-1}(x_1)(C(x_1, x_2)x_2 + G(x_1))$ are known non-linear functions; $w = M^{-1}(x_1)F_u$ denotes the disturbances and system uncertainties, which satisfies $|w| \leq b$ where b is a positive scalar; and $u \in \mathfrak{R}$ is the control input.

Through the application of Theorem 1, a standard predefined-time controller design method is presented here based on the sliding mode control technique for system (17). In the control framework, a new predefined-time sliding mode surface is first designed using piecewise function methods, in order to avoid potential singularities. A non-singular sliding mode controller is then designed, which ensures that the system (17) is predefined-time stable.

Choose a regulator function $\psi_1(\cdot) \in [a, b]$ which satisfies the sufficient conditions (i) and (ii) in Theorem 1, and define $V_1 = \frac{1}{2}x_1^2$. For the system (17), a novel non-singular sliding mode surface is designed as follows:

$$\begin{cases} s = x_2 + \frac{(b-a)V_1^q x_1 \Phi}{2p_1 T_1} \\ \Phi = \begin{cases} H_1 V_1^{-p_1 - q}, & V_1 \geq \eta_0 \\ H_1(k_1 V_1 + k_2 V_1^2), & V_1 < \eta_0 \end{cases} \end{cases} \tag{18}$$

where $0 < p_1 < \frac{1}{2}, q > 1, \eta_0 > 0$, and $H_1 = 1/h_1$ with $h_1 = \frac{d\psi_1}{dV_1^{p_1}}$; $T_1 > 0$ is a predefined-time constant; and $k_1 = 2\eta_0^{-1-p_1-q}$ and $k_2 = -\eta_0^{-2-p_1-q}$ are selected to meet the continuity of s . Differentiating s yields

$$\dot{s} = \dot{x}_2 + \frac{(b-a)(\dot{\Phi}V_1^q x_1 + x_2 V_1^q \dot{\Phi} + qV_1^{q-1} \dot{V}_1 \Phi x_1)}{2p_1 T_1}. \tag{19}$$

The terms $x_2 V_1^q \Phi$ and $qV_1^{q-1} \dot{V}_1 \Phi x_1$ are non-singular. Then, $\dot{\Phi}$ can be computed from (18), as follows:

$$\dot{\Phi} = \begin{cases} \dot{H}_1 V_1^{-p_1-q} + (-p_1 - q)V_1^{-p_1-q-1} \dot{V}_1 H_1, & V_1 \geq \eta_0 \\ \dot{H}_1(k_1 V_1 + k_2 V_1^2) + H_1(k_1 \dot{V}_1 + 2k_2 V_1 \dot{V}_1), & V_1 < \eta_0 \end{cases}, \tag{20}$$

where $\dot{H}_1 = -\frac{dh_1}{dV_1^{p_1}} p_1 V_1^{p_1-1} \dot{V}_1 / h_1^2$. Specifically, it can be observed that, when $V_1 \geq \eta_0 > 0$, $\dot{\Phi}$ will not exhibit singularity. When $V_1 < \eta_0$, it is possible for the term \dot{H}_1 to have a negative power term of x_1 . Then, according to the definition of V_1 , assuming the term $\frac{dh_1}{dV_1^{p_1}} p_1 V_1^{p_1-1} \dot{V}_1 = |x_1|^{-\eta} \chi(x_1, \dot{x}_1)$ with $\eta \geq 0$ and $\lim_{x_1 \rightarrow 0} \chi(x_1, \dot{x}_1) = 0$ is reasonable, $\chi(x_1, \dot{x}_1)$ denotes a smooth function. Hence, the limit of the term $\dot{H}_1 V_1^q x_1 = \lim_{x_1 \rightarrow 0} \frac{x_1 \chi(x_1, \dot{x}_1)}{2^q |x_1|^{1-(2q+1-\eta)}}$ exists when $2q + 1 - \eta > 1$. Therefore, there is no singularity for the term $\dot{\Phi} V_1^q x_1$ when an appropriate q is selected. For system (17), the control input u is designed as

$$u = g^{-1}(x_1, x_2) \left(-\frac{(b-a)(\dot{\Phi}V_1^q x_1 + x_2 V_1^q \dot{\Phi} + qV_1^{q-1} \dot{V}_1 \Phi x_1)}{2p_1 T_1} - \frac{(b-a)H_s s}{2T_2 p_1} V_2^{-p_1} - b\text{sign}(s) - f(x_1, x_2) \right), \tag{21}$$

where $H_s = 1/\frac{d\psi_1(V_2)}{dV_2^{p_1}}$ and $\kappa > 0$. $T_2 > 0$ is the predefined convergence time, and $V_2 = \frac{1}{2}s^2$ is a Lyapunov function. Summarizing the above analysis, the presented predefined-time sliding mode controller (21) has no singularity.

Theorem 2. *In view of the system (17), if the sliding mode surface is designed as (18) and the non-singular predefined-time sliding mode controller is designed as (21), the states x_1 and x_2 will converge to a small region around zero along the surface within a predefined-time $T = T_1 + T_2$.*

Proof. Taking the time derivative of V_2 and inserting (21) and (19), one has

$$\begin{aligned} \dot{V}_2 &= s(f(x_1, x_2) + g(x_1, x_2)u + w - \frac{(b-a)H_s s}{2T_2 p_1} V_2^{-p_1} - b\text{sign}(s)) \\ &\leq -\frac{b-a}{2H_s T_2 p_1} V_2^{1-p_1}. \end{aligned} \tag{22}$$

From (22), V_2 will converge to zero within the predefined time T_2 on the basis of Theorem 1. The ideal sliding mode motion is established simultaneously. Once s reaches zero, the designed sliding surface (18) satisfies $s = 0$. Then, one has $x_2 = -\frac{(b-a)H_1 x_1}{2p_1 T_1} V_1^{-p_1}$ when $|x_1| \geq \sqrt{2\eta_0}$. Taking the time derivative of V_1 , one can obtain $\dot{V}_1 = -\frac{H_1}{p_1 T_c} V_1^{1-p_1}$. Using Theorem 1, x_1 and x_2 will converge to the origin along the sliding manifold within the predefined time T_1 . When $|x_1| < \sqrt{2\eta_0}$ approaches zero along the general sliding manifold, one has $x_2 = -\frac{(b-a)V_1^q H_1 x_1}{2p_1 T_1} (k_1 V_1 + k_2 V_1^2)$. The second phase of s in (18) is asymptotically stable. Hence, the predefined-time convergence of system state x_1 —that

is, $\lim_{t \rightarrow (T_1+T_2)} |x_1| < \sqrt{2\eta_0}$ —is achieved, and the singularity problem of predefined-time sliding control can be circumvented.

Moreover, the presented Theorem 1 can also be applied for the stability analysis of high-order non-linear systems. In this case, the recursive sliding mode control scheme [37–39] or integral high-order sliding controller [40–42] can be designed to stabilize the high-order systems. In the sliding mode control method, the proposed Theorem 1 can be used for the controller design of the reaching phase. One can construct a controller to satisfy the condition (iii) of Theorem 1, ensuring that the system states reach the sliding mode surface within a predefined time. Once the ideal sliding mode is established, the system states will converge to the origin along the sliding mode surface. Therefore, the stability of high-order non-linear systems can be achieved. \square

5. Simulation Examples

To validate the previous theoretical results, the proposed predefined-time control framework (21) was applied to a second-order Lagrangian system and an actual attitude control system, and the effectiveness of the designed control scheme was verified through numerical simulation.

5.1. Predefined-Time Controller for a Second-Order Lagrangian System

Consider system (17) with $f(x_1, x_2) = (x_1 + 1)^2(x_2^2 + x_2 \sin(x_1))$, $g(x_1, x_2) = (x_1 + 1)^2$, and $w = \sin(t)(x_1 + 1)^2$. The control input is given by (21). In order to verify that, under different initial conditions, the designed predefined-time controllers can ensure that the state converges to the vicinity of the origin within a predefined time, Monte Carlo simulations with 500 dispersed scenarios were conducted. In the simulation, the initial conditions of x_1 and x_2 satisfied $x_1(0) \in [-1300, 1300]$ and $x_2(0) \in [-100, 100]$, respectively.

On the one hand, for the sliding surface (18), we selected a monotonically increasing regulator function $\psi_1(v) = b/(a + e^{-\alpha v^{p_1}})$ with $0 < p_1 < 1$, $\alpha > 0$, $a > 0$, and $b > 0$. One can obtain that $\psi_1(v) \in [\frac{b}{a+1}, \frac{b}{a}]$. Therefore, one has $H_1 = \frac{1}{b\alpha}(a + e^{-\alpha V_1^{p_1}})^2 e^{\alpha V_1^{p_1}} > 0$ and $H_s = \frac{1}{b\alpha}(a + e^{-\alpha V_2^{p_1}})^2 e^{\alpha V_2^{p_1}} > 0$. Thus, the sufficient conditions (i) and (ii) for the regulator function $\psi(\cdot)$ in Theorem 1 have been satisfied. On this basis, the specific controller of (21), guaranteeing controlled system predefined-time stability, can be obtained using the regulator function $\psi_1(v)$. In the Monte Carlo simulations, the control parameters were chosen as $a = 1$, $b = 3$, $\alpha = 1$, and $p_1 = 0.051$. The predefined-time constants were set as $T_1 = 0.5$ and $T_2 = 0.5$. Figure 1a,c display the convergence performance of the system state x_1 and sliding surface s driven by the controller (21). The convergence accuracy of system state x_1 was superior to 1×10^{-4} for any initial state, which can be verified from the depiction of the steady behavior of state $|x_1|$ in Figure 1b. To achieve predefined-time stability of the second-order non-linear system, the required control input is also illustrated in Figure 1d. The simulation results show that the system state converged to the origin within the predefined time $T_s = T_1 + T_2 = 1$ second.

On the other hand, for the sliding surface (18), we selected a monotonically decreasing regulator function $\psi_1(v) = \bar{a} + e^{-\bar{\alpha} v^{p_1}}$ with $0 < p_1 < 1$, $\bar{\alpha} > 0$, and $\bar{a} > 0$. One can obtain that $\psi_1(v) \in (\bar{a}, \bar{a} + 1]$. Therefore, one has $H_1 = -\frac{1}{\bar{\alpha}} e^{\bar{\alpha} V_1^{p_1}} < 0$ and $H_s = -\frac{1}{\bar{\alpha}} e^{\bar{\alpha} V_2^{p_1}} < 0$. The sufficient conditions (i) and (ii) for the regulator function $\psi(\cdot)$ in Corollary 1 were satisfied. Hence, the unique control input (21) with the decreasing regulator function $\psi_1(v)$ being used could be obtained to ensure the predefined-time stability of the controlled system. In the Monte Carlo simulations, the control gains were selected as $\bar{\alpha} = 1$ and $p_1 = 0.05$, and the given time constants were $T_1 = 0.5$ and $T_2 = 0.5$. Figure 2a,c plot the convergence performance of the system state x_1 and sliding surface s , respectively. From Figure 2b, it can be observed that, for any initial state value, the system state converged to a small region around zero (i.e., $|x_1| < 1 \times 10^{-3}$). The time response curve of the control input (21) is shown in Figure 2d. One can see that the system state x_1 converged to the small region around within the predefined time $T_s = T_1 + T_2 = 1$ second. Summarizing the analysis

of the above simulation results, the theoretical results in Theorem 2 were numerically validated through Monte Carlo simulations.

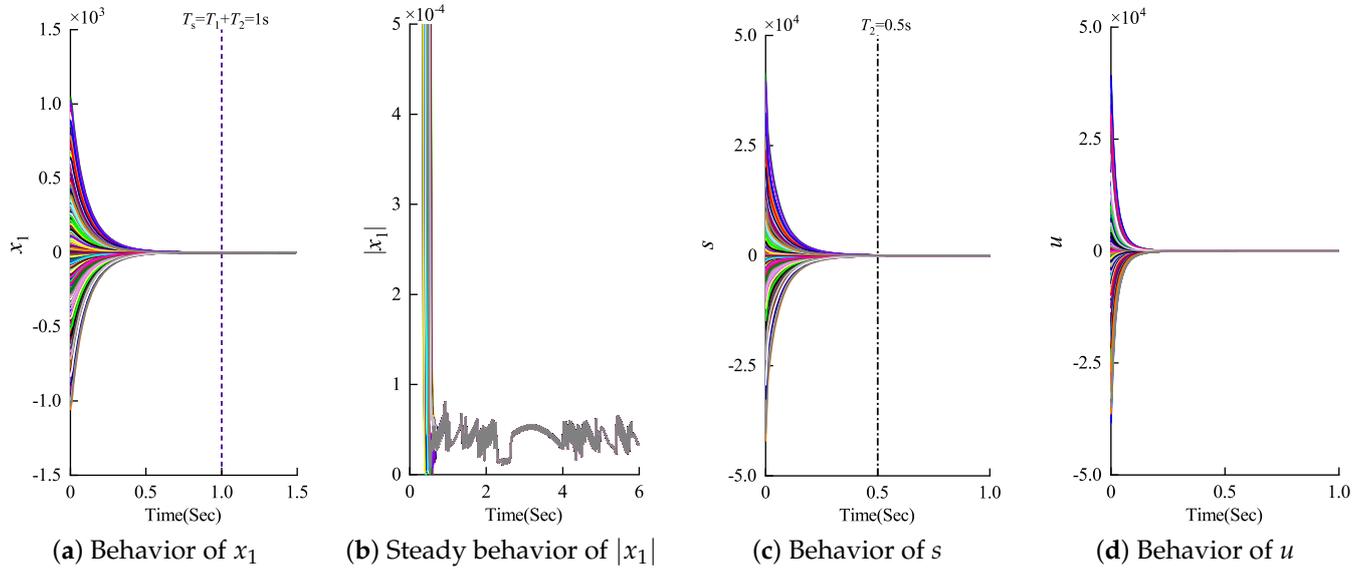


Figure 1. Convergence behavior of system state x_1 , sliding surface s , and control input u in the Monte Carlo simulations with an increasing function $\psi_1(v) = b/(a + e^{-\alpha v^{p_1}})$.

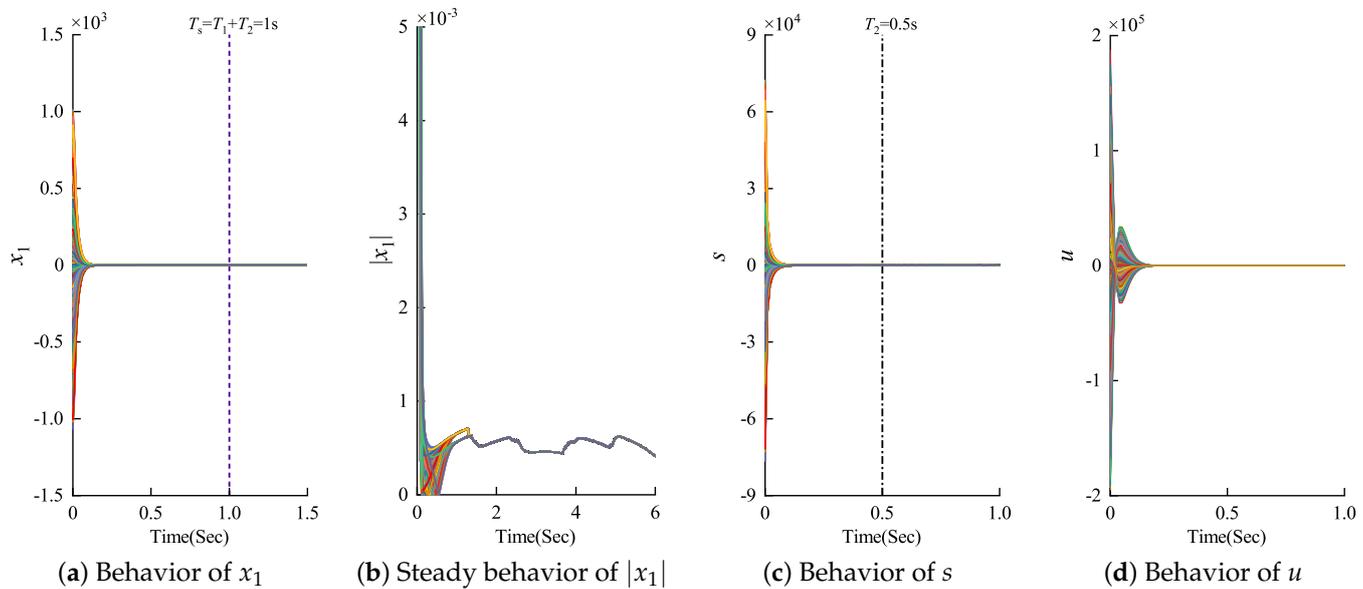


Figure 2. Convergence behavior of system state x_1 , sliding surface s , and control input u in the Monte Carlo simulations with a decreasing function $\psi_1(v) = \bar{a} + e^{-\bar{\alpha} v^{p_1}}$.

5.2. Predefined-Time Controller for a Spacecraft Attitude Control System

To verify the applicability and effectiveness of Theorem 2 in the context of actual control systems, a spacecraft attitude stabilization control system was taken as an example for verification. The MRPs $\rho = [\rho_1 \ \rho_2 \ \rho_3]^T$ were chosen to represent the attitude information. The attitude control model of a rigid spacecraft is given as follows [43]:

$$\begin{cases} \dot{\rho} = \Psi(\rho)\omega \\ J\dot{\omega} = -\omega^\times J\omega + u + u_d \end{cases} \quad (23)$$

with $\omega = [\omega_1 \ \omega_2 \ \omega_3]^\top$ being the angular velocity and $\omega^\times = [0 \ -\omega_3 \ \omega_2; \ \omega_3 \ 0 \ -\omega_1; \ -\omega_2 \ \omega_1 \ 0] \in \mathfrak{R}^{3 \times 3}$; $J \in \mathfrak{R}^{3 \times 3}$ is the inertia matrix of the spacecraft; $u \in \mathfrak{R}^{3 \times 1}$ denotes the control input; $u_d \in \mathfrak{R}^{3 \times 1}$ represents the external disturbance torque; $\Psi(\rho) \in \mathfrak{R}^{3 \times 3}$ is given as $\Psi(\rho) = \frac{(1-\rho^\top \rho)I_3 + 2\rho^\times + 2\rho\rho^\top}{4}$ with $\rho^\times = [0 \ -\rho_3 \ \rho_2; \ \rho_3 \ 0 \ -\rho_1; \ -\rho_2 \ \rho_1 \ 0] \in \mathfrak{R}^{3 \times 3}$; and $I_3 \in \mathfrak{R}^{3 \times 3}$ is an identity matrix. The system (23) can be transformed into

$$M\ddot{\rho} + C(\rho, \dot{\rho})\dot{\rho} = F^\top u + F^\top u_d, \tag{24}$$

with $C(\rho, \dot{\rho}) = -F^\top (JF\dot{\Psi} - (JF\dot{\rho})^\times)F$, $F = \Psi^{-1}$, and $M = F^\top JF$. Defining $x_1 = \rho$ and $x_2 = \dot{\rho}$, the system (24) can be rewritten as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = Px_2 + Qu + d \end{cases} \tag{25}$$

with $P = -M^{-1}C(x_1, x_2)$ and $Q = M^{-1}F^\top$, and $d = M^{-1}F^\top u_d$ denotes the synthetic external disturbance. Assume $\|d\| \leq b_1$, with b_1 being a positive real number.

To stabilize the attitude control system within a predefined time, the corresponding sliding mode surface and attitude controller were designed according to Theorem 2. Hence, the regulator function was designed as $\psi_a(V) = m \int_{-\frac{\ln n}{m}}^{z_1} e^{-mx} dx \in [n - 1, n]$ with $z_1 = \ln(V^{p_1} + 1)$, $0 < p_1 < 1$, $n > 1$, and $m > 0$. Referring to (18) and using the regulator function $\psi_a(V)$, a non-singular sliding mode surface for system (25) was developed as follows:

$$\begin{cases} S = x_2 + \frac{V_x^q x_1 \Phi_1}{2p_1 T_1} \\ \Phi_1 = \begin{cases} H_1 V_x^{-p_1 - q}, & V_x \geq \eta_0 \\ H_1 (k_1 V_x + k_2 V_x^2), & V_x < \eta_0 \end{cases} \end{cases} \tag{26}$$

with $V_x = \frac{1}{2}x_1^\top x_1$. $H_1 = 1/h_1$ with $h_1 = \frac{d\psi_a(V_x)}{dV_x^{p_1}} = m(V_x^{p_1} + 1)^{-m-1} > 0$. Using (26) and recalling (21), a non-singular predefined-time sliding mode controller for system (23) was designed, as follows:

$$u = Q^{-1} \left(-\frac{(\dot{\Phi}_1 V_x^q x_1 + x_2 V_x^q \Phi_1 + q V_x^{q-1} \dot{V}_x \Phi_1 x_1)}{2p_1 T_1} - \frac{H_s S}{2T_2 p_1} V_s^{-p_1} - b_1 \text{sign}(S) - Px_2 \right), \tag{27}$$

with $V_s = \frac{1}{2}S^\top S$. $H_s = 1/h_s$ with $h_s = \frac{d\psi_a(V_s)}{dV_s^{p_1}} = m(V_s^{p_1} + 1)^{-m-1} > 0$.

Then, a simulation study was conducted to test the control performance of the designed attitude controller (27). In the simulation, the inertia matrix of the spacecraft was set as $J = [20 \ 0 \ 0.9; \ 0 \ 17 \ 0; \ 0.9 \ 0 \ 15] \text{kg} \cdot \text{m}^2$. The external disturbance was assumed as $u_d = 0.01[\sin(0.5t) \ \cos(0.5t) \ \sin(0.4t)]^\top \text{Nm}$. The initial settings of the attitude control system were set as $\rho(0) = [1.0 \ -0.5 \ -1.5]^\top$ and $\omega(0) = [0.14 \ -0.11 \ 0.06]^\top \text{rad/s}$. The main control gains in (26) and (27) were selected as $T_1 = 5$, $T_2 = 10$, $p_1 = 0.1$, $q = 2$, $m = 2$, $n = 3$, $b_1 = 0.02$, and $\eta_0 = 0.02$. According to the simulation results, the convergence performance of spacecraft attitude MRPs and sliding mode surface are shown in Figure 3a and Figure 3b, respectively. The attitude MRPs ρ converged to a small region around the origin rapidly within the predefined time of $T = T_1 + T_2 = 15$ s. To achieve attitude stabilization, the time response curve of the control input is plotted in Figure 3c. It can be observed that attitude stabilization was completed when $t \geq T_1 + T_2 = 15$ s, and the control input remained around zero. Hence, the conclusions in Theorem 2 were numerically validated.

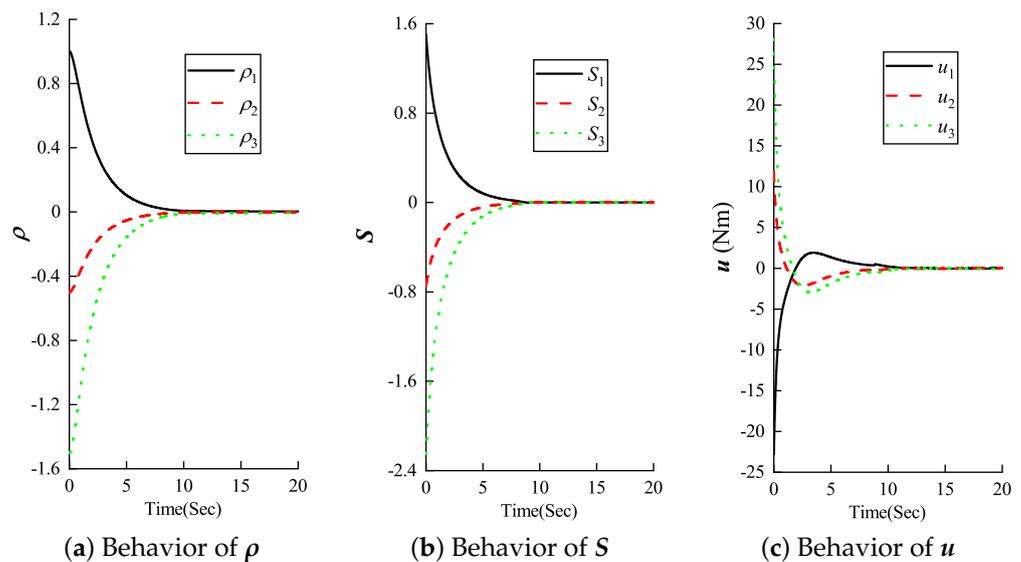


Figure 3. Time response curve of MRPs ρ , sliding surface S , and control input u in attitude stabilization simulation example.

6. Conclusions

This study investigated sufficient conditions within the framework of Lyapunov theory for guaranteeing the predefined-/finite-time stability of non-linear systems. The developed Lyapunov-based theorem allowed us to establish equivalence with existing Lyapunov-based theorems for predefined-/finite-time stability in non-linear systems. The proposed theorem not only allows for the establishment of more possible Lyapunov-based predefined-/finite-time stability theorems through the choice of different regulator functions but also covers existing Lyapunov predefined-/finite-time stability theorems [6,19,23,26,28,29]. On this basis, a standard non-singular sliding mode control framework guaranteeing the predefined-time stability of second-order Lagrangian systems was provided. Furthermore, numerical simulation results verified the effectiveness of the proposed control approaches.

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