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Well-Posedness Results of Certain Variational Inequalities

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Abstract: Well-posedness and generalized well-posedness results are examined for a class of commanded variational inequality problems. In this regard, by using the concepts of hemicontinuity, monotonicity, and pseudomonotonicity of the considered functional, and by introducing the set of approximating solutions of the considered commanded variational inequality problems, we establish several well-posedness and generalized well-posedness results. Moreover, some illustrative examples are provided to highlight the effectiveness of the results obtained in the paper.

Keywords: well-posedness and generalized well-posedness; commanded variational inequality; monotonicity; hemicontinuity; pseudomonotonicity; functional

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1. Introduction

It is well-known that the theory of variational inequalities comes from the calculus of variations. Initially, this theory was developed to investigate some equilibrium problems. Variational inequalities in finite and infinite dimensions have been widely considered as a mathematical tool for investigating partial differential equations, having many applications principally in mechanics, and optimization problems that arise in economics, finance, or game theory, (see, for instance, [1–4]).

In many situations, solving some optimization problems with the classical methods (see [5]) becomes very complicated and, moreover, these methods may not always ensure the existence of exact solutions. In such cases, the concept of (Tykhonov) well-posedness associated with the considered problem ensures the convergence for the sequence of approximating solutions toward the exact solution. Generally speaking, the well-posedness concept represents an important technique to investigate the related problems, such as: fixed point problems [6], variational inequalities [7], hemivariational inequalities [8–12], complementary problems [13], equilibrium problems [14], Nash equilibrium problems [15], etc. Let us mention that the notion of well-posedness for optimization problems without constraints was introduced by Tykhonov [16]. Since then, different types of well-posedness for variational inequalities were considered, for example, Levitin-Polyak well-posedness [17], and generic well-posedness [18–22]. Ceng and Yao [23] studied the generalized well-posedness of a mixed variational inequality and proved that the generalized well-posedness for the inequality problem is equivalent to that of fixed point problems and inclusion problems. For other different but connected ideas to this topic, the reader is directed to [24–33].

Further, the hemivariational inequality, as a generalization of a variational inequality, was studied by Panagiotopoulos [34]. The well-posedness for hemivariational inequalities was analyzed by Goeleven and Mentagui [35]. Thereafter, Xiao et al. [12,36,37] investigated the well-posedness for hemivariational inequalities by introducing the approximating sequences and establishing some metric characterizations in Euclidean spaces. Recently, Hu et al. [38] obtained certain equivalence results for well-posedness associated to split variational-hemivariational inequality. Also, Bai et al. [39] studied a class of generalized mixed hemivariational-variational inequalities of elliptic type in a Banach space and obtained a well-posedness result for the inequality, including existence, uniqueness, and stability of the solution. Moreover, the well-posedness of history and state-dependent sweeping processes has been investigated by several researchers ([40–42]).

The multi-time (or multi-dimensional) optimal control theory, which is in close connection with the calculus of variations, solves different kinds of operations research problems that arise in applied science or technology. In the last decade, this theory was intensively considered both theoretically and practically ([43–47]). Therefore, variational inequality with multiple variables of evolution represents an interesting generalization of variational inequality (see [48–51]), with many real applications provided by the involved functionals.

In this study, motivated by the aforementioned research works, we study the well-posedness and generalized well-posedness for a class of commanded variational inequalities governed by multiple integral functionals. Concretely, by using the hemicontinuity, monotonicity, and pseudomonotonicity associated with the considered multiple integral functional, and by introducing the set of approximating solutions for the considered class of controlled variational inequalities, we establish several characterization results on well-posedness and well-posedness in the generalized sense for the inequality. Next, let's highlight the main merits of this paper. Firstly, most of the former research papers have been investigated in classical spaces with finite dimensions. In this paper, the mathematical context is defined by some function spaces with infinite dimensions and controlled functionals of multiple integral types. Recently, Treanţă [45] studied some variational inequality-constrained control problems (that is, some optimization problems with controlled variational inequalities as constraints), which imply partial derivatives of second-order. Also, the curvilinear case for controlled variational inequality problem was investigated in Treanţă [48]. Moreover, by considering the functional (variational) derivative, well-posed isoperimetric-type constrained variational control problems have been studied in Treanţă [51]. In consequence, this paper deals with a special situation in which the variational problem is a controlled variational inequality defined by functionals of multiple integral types.

The current paper is organized as follows. In Section 2, we present the monotonicity, hemicontinuity, and pseudomonotonicity for a multiple integral functional. In Section 3, by introducing the approximating solution set of the considered commanded variational inequalities, we formulate the notions of well-posedness and generalized well-posedness associated with this class of inequalities. Then, we prove that well-posedness can be studied in the terms of existence and uniqueness of the solution. Moreover, we state sufficient conditions for the generalized well-posedness by assuming the boundedness of approximate solutions. The results stated in this study are illustrated with some examples. In Section 4, the paper ends with some conclusions.

2. Problem Formulation and Preliminaries

Let K be a compact set in $\mathbb{R}^m$ and consider $\tau = (\tau^\gamma) \in K$, $\gamma \in \{1, \ldots, m\}$. Also consider $\mathcal{P}$ is the space of piece-wise smooth *state* functions $s \colon K \to \mathbb{R}^n$, having the norm

$$\| s \| = \| s \|_\infty + \sum_{\gamma=1}^{m} \| s_\gamma \|_\infty, \quad \forall s \in \mathcal{P},$$

where we used the notation $s_\gamma := \dfrac{\partial s}{\partial \tau^\gamma}$, $\gamma \in \{1, \ldots, m\}$. Denote by $\mathcal{Q}$ the space consisting of piece-wise continuous *control* functions $u \colon K \to \mathbb{R}^k$, having the uniform norm.

In the following, we assume that $P \times Q$ is a closed, convex and nonempty subset of $\mathcal{P} \times \mathcal{Q}$, with $(s,u)|_{\partial K} =$ given, $\frac{\partial s^i}{\partial \tau^\gamma} = X^i_\gamma(\tau, s, u) =$ given, with the inner product

$$\langle (s,u),(z,w)\rangle = \int_K \Big[s(\tau)\cdot z(\tau) + u(\tau)\cdot w(\tau)\Big] d\tau$$

$$= \int_K \Big[\sum_{i=1}^{n} s^i(\tau) z^i(\tau) + \sum_{j=1}^{k} u^j(\tau) w^j(\tau)\Big] d\tau, \quad \forall (s,u),(z,w) \in \mathcal{P} \times \mathcal{Q}$$

and the induced norm, where $d\tau = d\tau^1 \cdots d\tau^m$ denotes the volume element on $\mathbb{R}^m$.

Let $J^1(\mathbb{R}^m, \mathbb{R}^n)$ be the first-order jet bundle associated with $\mathbb{R}^m$ and $\mathbb{R}^n$. By considering the real-valued continuously differentiable function $f : J^1(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^k \to \mathbb{R}$, we define the following scalar functional governed by a multiple integral:

$$F : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}, \quad F(s,u) = \int_K f(\tau, s, s_\gamma, u) d\tau,$$

where $s_\gamma = \dfrac{\partial s}{\partial \tau^\gamma}$, $\gamma \in \{1, \ldots, m\}$.

Next, we introduce the commanded variational inequality (in short, CVI): find $(s,u) \in P \times Q$ such that

$$\int_K \left[\frac{\partial f}{\partial s}(\pi_{s,u})(z-s) + \frac{\partial f}{\partial s_\gamma}(\pi_{s,u}) D_\gamma (z-s)\right] d\tau$$
$$+ \int_K \left[\frac{\partial f}{\partial u}(\pi_{s,u})(w-u)\right] d\tau \geq 0, \quad \forall (z,w) \in P \times Q,$$

where D_γ is the total derivative operator and $(\pi_{s,u}) := (\tau, s, s_\gamma, u)$.

Let $\mathcal{S}$ be the feasible solution set of (CVI),

$$\begin{aligned}\mathcal{S} = \Big\{ (s,u) \in P \times Q : \int_K \Big[&(z(\tau) - s(\tau)) \frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) \\ &+ D_\gamma (z(\tau) - s(\tau)) \frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau)) \\ &+ (w(\tau) - u(\tau)) \frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) \Big] d\tau \geq 0, \\ &\forall (z,w) \in P \times Q \Big\},\end{aligned}$$

where $(\pi_{s,u}(\tau)) := (\tau, s(\tau), s_\gamma(\tau), u(\tau))$.

Definition 1. *We say the functional* $\int_K f(\pi_{s,u}(\tau)) d\tau$ *is monotone on* $P \times Q$ *if, for any* $(s,u),(z,w) \in P \times Q$*, the following inequality holds:*

$$\begin{aligned}\int_K \Big[&(s(\tau) - z(\tau)) \left(\frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) \right) \\ &+ (u(\tau) - w(\tau)) \left(\frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \right) \\ &+ D_\gamma (s(\tau) - z(\tau)) \left(\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau)) \right) \Big] d\tau \geq 0.\end{aligned}$$

Example 1. *Let* $m = 2$ *and* $K = [0,1]^2$*. Consider*

$$f(\pi_{s,u}(\tau)) = u(\tau) + e^{s(\tau)} - 1.$$

Then, we show that $\int_K f(\pi_{s,u}(\tau))d\tau$ *is monotone on* $P \times Q = C^1(K, \mathbb{R}) \times C(K, \mathbb{R})$*. Indeed, we get*

$$\begin{aligned}
&\int_K \Big[(s(\tau) - z(\tau))\left(\frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau))\right) \\
&\qquad + (u(\tau) - w(\tau))\left(\frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial u}(\pi_{z,w}(\tau))\right) \\
&\qquad + D_\gamma(s(\tau) - z(\tau))\left(\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\right)\Big]d\tau \\
&= \int_K (s(\tau) - z(\tau))(e^{s(\tau)} - e^{z(\tau)})d\tau \geq 0, \ \forall (s,u),\ (z,w) \in P \times Q.
\end{aligned}$$

Definition 2. *We say the functional* $\int_K f(\pi_{s,u}(\tau))d\tau$ *is pseudomonotone on* $P \times Q$ *if, for all* $(s,u),\ (z,w) \in P \times Q$*, the following implication holds:*

$$\begin{aligned}
&\int_K \Big[(s(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (u(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&\qquad + D_\gamma(s(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0 \\
\Rightarrow\ &\int_K \Big[(s(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) + (u(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) \\
&\qquad + D_\gamma(s(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau))\Big]d\tau \geq 0.
\end{aligned}$$

Example 2. *Let* $m = 2$ *and* $K = [0,1]^2$*. Consider*

$$f(\pi_{s,u}(\tau)) = \sin u(\tau) + s(\tau)e^{s(\tau)}.$$

Then, we show that the functional $\int_K f(\pi_{s,u}(\tau))d\tau$ *is pseudomonotone on*

$$P \times Q = C^1(K, [-1,1]) \times C(K, [-1,1].$$

We obtain

$$\begin{aligned}
&\int_K \Big[(s(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (u(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&\qquad + D_\gamma(s(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \\
&= \int_K \Big[(u(\tau) - w(\tau))\cos w(\tau) + (s(\tau) - z(\tau))(e^{z(\tau)} + z(\tau)e^{z(\tau)})\Big]d\tau \geq 0 \\
&\qquad \forall (s,u),\ (z,w) \in P \times Q \\
\Rightarrow\ &\int_K \Big[(s(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) + (u(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) \\
&\qquad + D_\gamma(s(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau))\Big]d\tau
\end{aligned}$$

$$= \int_K \Big[(u(\tau) - w(\tau)) \cos u(\tau) + (s(\tau) - z(\tau))(e^{s(\tau)} + s(\tau)e^{s(\tau)})\Big] d\tau \geq 0$$
$$\forall (s, u),\ (z, w) \in P \times Q.$$

But, it is not monotone on $P \times Q$, because

$$\int_K \Big[(s(\tau) - z(\tau))\left(\frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau))\right)$$
$$+ (u(\tau) - w(\tau))\left(\frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial u}(\pi_{z,w}(\tau))\right)$$
$$+ D_\gamma(s(\tau) - z(\tau))\left(\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau)) - \frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\right)\Big] d\tau$$
$$= \int_K \Big[(u(\tau) - w(\tau))(\cos u(\tau) - \cos w(\tau))$$
$$+ (s(\tau) - z(\tau))(s(\tau)e^{s(\tau)} + e^{s(\tau)} - z(\tau)e^{z(\tau)} - e^{z(\tau)})\Big] d\tau \ngeq 0,$$
$$\forall (s, u),\ (z, w) \in P \times Q.$$

By considering Usman and Khan [52], we formulate the following definition of hemicontinuity for the aforementioned multiple integral functional.

Definition 3. *The functional $\int_K f(\pi_{s,u}(\tau)) d\tau$ is said to be hemicontinuous on $P \times Q$ if, for all $(s, u), (z, w) \in P \times Q$, the application*

$$\sigma \to \left\langle ((s(\tau), u(\tau)) - (z(\tau), w(\tau)), \left(\frac{\delta F}{\delta s_\sigma}, \frac{\delta F}{\delta u_\sigma}\right)\right\rangle, \quad 0 \leq \sigma \leq 1$$

is continuous at 0^+, where

$$\frac{\delta F}{\delta s_\sigma} := \frac{\partial f}{\partial s}(\pi_{s_\sigma, u_\sigma}(\tau)) - D_\gamma \frac{\partial f}{\partial s_\gamma}(\pi_{s_\sigma, u_\sigma}(\tau)) \in P,$$

$$\frac{\delta F}{\delta u_\sigma} := \frac{\partial f}{\partial u}(\pi_{s_\sigma, u_\sigma}(\tau)) \in Q,$$
$$s_\sigma := \sigma s + (1 - \sigma) z, \quad u_\sigma := \sigma u + (1 - \sigma) w.$$

Lemma 1. *Consider the functional $\int_K f(\pi_{s,u}(\tau)) d\tau$ is hemicontinuous and pseudomonotone on the closed, convex and nonempty set $P \times Q$. Then, $(s, u) \in P \times Q$ solves (CVI) if and only if it solves*

$$\int_K \Big[(z(\tau) - s(\tau)) \frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u(\tau)) \frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+ D_\gamma(z(\tau) - s(\tau)) \frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big] d\tau \geq 0, \quad \forall (z, w) \in P \times Q.$$

Proof. Suppose the pair $(s, u) \in P \times Q$ is solution for (CVI). In consequence, it implies

$$\int_K \Big[(z(\tau) - s(\tau)) \frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) + (w(\tau) - u(\tau)) \frac{\partial f}{\partial u}(\pi_{s,u}(\tau))$$
$$+ D_\gamma(z(\tau) - s(\tau)) \frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau))\Big] d\tau \geq 0, \quad \forall (z, w) \in P \times Q.$$

By considering the pseudomonotonicity of $\int_K f(\pi_{s,u}(\tau))d\tau$, it follows

$$\int_K \Big[(z(\tau)-s(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau))+(w(\tau)-u(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+D_\gamma(z(\tau)-s(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w)\in P\times Q.$$

Conversely, assume that

$$\int_K \Big[(z(\tau)-s(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau))+(w(\tau)-u(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+D_\gamma(z(\tau)-s(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w)\in P\times Q.$$

Further, for $\sigma \in (0,1)$ and $(z,w)\in P\times Q$, consider

$$(z_\sigma, w_\sigma) = ((1-\sigma)s+\sigma z, (1-\sigma)u+\sigma w)\in P\times Q.$$

The above inequality implies

$$\int_K \Big[(z_\sigma(\tau)-s(\tau))\frac{\partial f}{\partial s}(\pi_{z_\sigma,w_\sigma}(\tau))+(w_\sigma(\tau)-u(\tau))\frac{\partial f}{\partial u}(\pi_{z_\sigma,w_\sigma}(\tau))$$
$$+D_\gamma(z_\sigma(\tau)-s(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z_\sigma,w_\sigma}(\tau))\Big]d\tau \geq 0, \quad (z,w)\in P\times Q,$$

and for $\sigma \to 0$ (by using the hemicontinuity of $\int_K f(\pi_{s,u}(\tau))d\tau$), we get

$$\int_K \Big[(z(\tau)-s(\tau))\frac{\partial f}{\partial s}(\pi_{s,u}(\tau))+(w(\tau)-u(\tau))\frac{\partial f}{\partial u}(\pi_{s,u}(\tau))$$
$$+D_\gamma(z(\tau)-s(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w)\in P\times Q,$$

which proves that $(s(\tau),u(\tau))$ solves (CVI). $\square$

3. Well-Posedness and Generalized Well-Posedness of (CVI)

In this section, well-posedness and generalized well-posedness are analyzed for the considered commanded variational inequalities.

Definition 4. *We say that a sequence $\{(s_n,u_n)\}\subset P\times Q$ is an approximating sequence for (CVI) if there exists a sequence of positive real numbers $\theta_n \to 0$ as $n\to\infty$, satisfying*

$$\int_K \Big[(z(\tau)-s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau))+(w(\tau)-u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau))$$
$$+D_\gamma(z(\tau)-s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau+\theta_n \geq 0, \quad \forall (z,w)\in P\times Q.$$

Definition 5. *The commanded variational inequality problem (CVI) is named well-posed if:*
(i) it has a unique solution $(s_0(\tau),u_0(\tau))$;
(ii) every approximating sequence of (CVI) converges to the unique solution $(s_0(\tau),u_0(\tau))$.

Definition 6. *The commanded variational inequality problem (CVI) is named generalized well-posed if:*

(i) the set of solutions of (CVI) is nonempty, that is, $\mathcal{S} \neq \emptyset$;
(ii) every approximating sequence of (CVI) has a subsequence that converges to some point of $\mathcal{S}$.

Let $\theta > 0$ be fixed. Now, for investigating well-posedness and generalized well-posedness for (CVI), we formulate the *approximating solution set* for (CVI), as follows:

$$\mathcal{S}_\theta = \Big\{(s,u) \in P \times Q : \int_K \Big[(z(\tau) - s(\tau))\frac{\partial f}{\partial s}(\pi_{s,u}(\tau)) + (w(\tau) - u(\tau))\frac{\partial f}{\partial u}(\pi_{s,u}(\tau)) + D_\gamma(z(\tau) - s(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s,u}(\tau))\Big]d\tau + \theta \geq 0,\ \forall(z,w) \in P \times Q\Big\}.$$

Remark 1. *Obviously, $\mathcal{S} = \mathcal{S}_\theta$, when $\theta = 0$, and it holds*

$$\mathcal{S} \subseteq \mathcal{S}_\theta \text{ for each } \theta > 0 \text{ and } \mathcal{S}_\theta \subset \mathcal{S}_\eta \text{ for } 0 < \theta \leq \eta.$$

Next, we define the diameter of B as follows:

$$diam\ B = \sup_{x,y \in B} \|x - y\|.$$

Theorem 1. *Let the functional $\int_K f(\pi_{s,u}(\tau))d\tau$ be hemicontinuous and monotone on the closed, convex and nonempty set $P \times Q$. Then the problem (CVI) is well-posed if and only if*

$$\mathcal{S}_\theta \neq \emptyset \text{ for all } \theta > 0 \text{ and } diam\ \mathcal{S}_\theta \to 0 \text{ as } \theta \to 0.$$

Proof. Assume that the problem (CVI) is well-posed. Then it has a unique solution $\mathcal{S} = \{(\bar{s}(\tau), \bar{u}(\tau))\}$. Since $\mathcal{S} \subseteq \mathcal{S}_\theta$, $\forall\theta > 0$, we get $\mathcal{S}_\theta \neq \emptyset$ for all $\theta > 0$. Consider, contrary to the result, that diam $\mathcal{S}_\theta \nrightarrow 0$ as $\theta \to 0$. Then there exist $r > 0$, a positive integer m, $\theta_n > 0$ with $\theta_n \to 0$ and $(s_n(\tau), u_n(\tau))$, $(s'_n(\tau), u'_n(\tau)) \in \mathcal{S}_{\theta_n}$ such that

$$\|(s_n(\tau), u_n(\tau)) - (s'_n(\tau), u'_n(\tau))\| > r, \quad \forall n \geq m. \tag{1}$$

Since $(s_n(\tau), u_n(\tau))$, $(s'_n(\tau), u'_n(\tau)) \in \mathcal{S}_{\theta_n}$, we get

$$\int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau)) + D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau + \theta_n \geq 0, \quad \forall(z,w) \in P \times Q$$

and

$$\int_K \Big[(z(\tau) - s'_n(\tau))\frac{\partial f}{\partial s}(\pi_{s'_n,u'_n}(\tau)) + (w(\tau) - u'_n(\tau))\frac{\partial f}{\partial u}(\pi_{s'_n,u'_n}(\tau)) + D_\gamma(z(\tau) - s'_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s'_n,u'_n}(\tau))\Big]d\tau + \theta_n \geq 0, \quad \forall(z,w) \in P \times Q.$$

Now, it is obvious that $\{(s_n(\tau), u_n(\tau))\}$ and $\{(s'_n(\tau), u'_n(\tau))\}$ are approximating sequences for (CVI). Moreover, they converge to the unique solution $(\bar{s}(\tau), \bar{u}(\tau))$ (by assumption, the problem (CVI) is well-posed). By computation, we obtain

$$\begin{aligned}
&\|(s_n(\tau), u_n(\tau)) - (s'_n(\tau), u'_n(\tau))\| \\
&= \|(s_n(\tau), u_n(\tau)) - (\bar{s}(\tau), \bar{u}(\tau)) + (\bar{s}(\tau), \bar{u}(\tau)) - (s'_n(\tau), u'_n(\tau))\| \\
&\leq \|(s_n(\tau), u_n(\tau)) - (\bar{s}(\tau), \bar{u}(\tau))\| + \|(\bar{s}(\tau), \bar{u}(\tau)) - (s'_n(\tau), u'_n(\tau))\| \leq \theta,
\end{aligned}$$

which contradicts (1), for some $\theta = r$.

Conversely, consider $\{(s_n(\tau), u_n(\tau))\}$ is an approximating sequence of (CVI). Consequently, there exists a sequence of positive real numbers $\theta_n \to 0$ as $n \to \infty$ such that

$$\int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau))$$

$$+ D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau + \theta_n \geq 0, \quad \forall (z, w) \in P \times Q \tag{2}$$

is fulfilled, which implies $(s_n(\tau), u_n(\tau)) \in \mathcal{S}_{\theta_n}$. Since diam $\mathcal{S}_{\theta_n} \to 0$ as $\theta_n \to 0$, we get $\{(s_n(\tau), u_n(\tau))\}$ is a Cauchy sequence converging to some point $(\bar{s}, \bar{u}) \in P \times Q$ (as $P \times Q$ is closed).

Since the functional $\int_K f(\pi_{s,u}(\tau))d\tau$ is monotone on $P \times Q$, for $(\bar{s}, \bar{u}),\ (z, w) \in P \times Q$, we get

$$\int_K \Big[(\bar{s}(\tau) - z(\tau))\left(\frac{\partial f}{\partial s}(\pi_{\bar{s},\bar{u}}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau))\right)$$

$$+(\bar{u}(\tau) - w(\tau))\left(\frac{\partial f}{\partial u}(\pi_{\bar{s},\bar{u}}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau))\right)$$

$$+D_\gamma(\bar{s}(\tau) - z(\tau))\left(\frac{\partial f}{\partial s_\gamma}(\pi_{\bar{s},\bar{u}}(\tau)) - \frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\right)\Big]d\tau \geq 0$$

namely,

$$\int_K \Big[(\bar{s}(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{\bar{s},\bar{u}}(\tau)) + (\bar{u}(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{\bar{s},\bar{u}}(\tau))$$

$$+D_\gamma(\bar{s}(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{\bar{s},\bar{u}}(\tau))\Big]d\tau$$

$$\geq \int_K \Big[(\bar{s}(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (\bar{u}(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$

$$+ D_\gamma(\bar{s}(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau. \tag{3}$$

By considering the limit as $n \to \infty$ in (2), it yields

$$\int_K \Big[(\bar{s}(\tau) - z(\tau))\frac{\partial f}{\partial s}(\pi_{\bar{s},\bar{u}}(\tau)) + (\bar{u}(\tau) - w(\tau))\frac{\partial f}{\partial u}(\pi_{\bar{s},\bar{u}}(\tau))$$

$$+ D_\gamma(\bar{s}(\tau) - z(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{\bar{s},\bar{u}}(\tau))\Big]d\tau \leq 0. \tag{4}$$

It follows from (3) and (4) that

$$\int_K \Big[(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - \bar{u}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$

$$+D_\gamma(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0.$$

By using Lemma 1, we obtain

$$\int_K \Big[(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s}(\pi_{\bar{s},\bar{u}}(\tau)) + (w(\tau) - \bar{u}(\tau))\frac{\partial f}{\partial u}(\pi_{\bar{s},\bar{u}}(\tau))$$

$$+D_\gamma(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{\bar{s},\bar{u}}(\tau))\Big]d\tau \geq 0,$$

which implies that $(\bar{s}(\tau), \bar{u}(\tau)) \in \mathcal{S}$. Let us prove the uniqueness of (CVI). Contrarily, suppose $(s_1(\tau), u_1(\tau)), (s_2(\tau), u_2(\tau))$ are two distinct solutions of (CVI). Then

$$0 < \|(s_1(\tau), u_1(\tau)) - (s_2(\tau), u_2(\tau))\| \leq \text{diam } \mathcal{S}_\theta \to 0 \text{ as } \theta \to 0,$$

and this completes the proof. □

Corollary 1. *Consider all the hypotheses of Theorem 1 are fulfilled. Then the controlled variational inequality (CVI) is well-posed if and only if*

$$\mathcal{S} \neq \emptyset \text{ and diam } \mathcal{S}_\theta \to 0 \text{ as } \theta \to 0.$$

Proof. The proof follows in the same manner as in Theorem 1. Hence, it is omitted. □

Theorem 2. *Let the functional* $\int_K f(\pi_{s,u}(\tau))d\tau$ *be hemicontinuous and monotone on the closed, convex and nonempty set* $P \times Q$*. Then (CVI) is well-posed if and only if it admits a unique solution.*

Proof. Let us consider that (CVI) is well-posed. Thus, it has a unique solution $(s_0(\tau), u_0(\tau))$. Conversely, consider that (CVI) has a unique solution $(s_0(\tau), u_0(\tau))$, but it is not well-posed. Consequently, there exists an approximating sequence $\{(s_n(\tau), u_n(\tau))\}$ of (CVI) which does not converge to $(s_0(\tau), u_0(\tau))$. Since $\{(s_n(\tau), u_n(\tau))\}$ is an approximating sequence of (CVI), there must exist a sequence of positive real numbers $\{\theta_n\}$ with $\theta_n \to 0$ as $n \to \infty$ such that

$$\int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau))$$
$$+ D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau + \theta_n \geq 0, \quad \forall (z, w) \in P \times Q. \tag{5}$$

In the following, we start by reductio ad absurdum to prove the boundedness of $\{(s_n(\tau), u_n(\tau))\}$. Suppose $\{(s_n(\tau), u_n(\tau))\}$ is not bounded, involving $\|(s_n(\tau), u_n(\tau))\| \to +\infty$ as $n \to +\infty$. Now, we consider $\delta_n(\tau) = \dfrac{1}{\|(s_n(\tau), u_n(\tau)) - (s_0(\tau), u_0(\tau))\|}$ and $(\mathsf{s}_n(\tau), \mathsf{u}_n(\tau)) = (s_0(\tau), u_0(\tau)) + \delta_n(\tau)[(s_n(\tau), u_n(\tau)) - (s_0(\tau), u_0(\tau))]$.

We can see that $\{(\mathsf{s}_n(\tau), \mathsf{u}_n(\tau))\}$ is bounded in $P \times Q$. So, passing to a subsequence if necessary, we may assume that

$$(\mathsf{s}_n(\tau), \mathsf{u}_n(\tau)) \to (\mathsf{s}(\tau), \mathsf{u}(\tau)) \text{ weakly in } P \times Q \neq (s_0(\tau), u_0(\tau)).$$

It is easy to verify that $(\mathsf{s}(\tau), \mathsf{u}(\tau)) \neq (s_0(\tau), u_0(\tau))$ thanks to $\|\delta_n(\tau)[(s_n(\tau), u_n(\tau)) - (s_0(\tau), u_0(\tau))\| = 1$ for all $n \in \mathbb{N}$. Since $(s_0(\tau), u_0(\tau))$ is a solution of (CVI),

$$\int_K \Big[(z(\tau) - s_0(\tau))\frac{\partial f}{\partial s}(\pi_{s_0,u_0}(\tau)) + (w(\tau) - u_0(\tau))\frac{\partial f}{\partial u}(\pi_{s_0,u_0}(\tau))$$
$$+D_\gamma(z(\tau) - s_0(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_0,u_0}(\tau))\Big]d\tau \geq 0, \quad \forall (z, w) \in P \times Q.$$

By considering Lemma 1, we obtain

$$\int_K \Big[(z(\tau) - s_0(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_0(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+ D_\gamma(z(\tau) - s_0(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0, \quad \forall (z, w) \in P \times Q. \tag{6}$$

Since the functional $\int_K f(\pi_{s,u}(\tau))d\tau$ is monotone on $P \times Q$, for $(s_n, u_n), (z, w) \in P \times Q$, we get

$$\int_K \Big[(s_n(\tau) - z(\tau))\Big(\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau))\Big)$$
$$+(u_n(\tau) - w(\tau))\Big(\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau)) - \frac{\partial f}{\partial s}(\pi_{z,w}(\tau))\Big)$$
$$+D_\gamma(s_n(\tau) - z(\tau))\Big(\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau)) - \frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big)\Big]d\tau \geq 0,$$

that is,

$$\begin{aligned}
&\int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau)) \\
&\qquad +D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau \\
&\leq \int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&\qquad + D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau.
\end{aligned} \tag{7}$$

Combining with (5) and (7), we have

$$\begin{aligned}
&\int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&+D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \\
&\geq -\theta_n \quad \forall (z, w) \in P \times Q.
\end{aligned}$$

Since $\delta_n \to 0$ as $n \to \infty$ (see $\{(s_n(\tau), u_n(\tau))\}$ is not bounded), we can consider $n_0 \in \mathbb{N}$ is large enough with $\delta_n < 1$, for $n \geq n_0$. Multiplying the above inequality and (6) by $\delta_n(\tau) > 0$ and $1 - \delta_n(\tau) > 0$, respectively, and making the summation, it implies

$$\begin{aligned}
&\int_K \Big[(z(\tau) - \mathsf{s}_n(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - \mathsf{u}_n(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&+D_\gamma(z(\tau) - \mathsf{s}_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \\
&\geq -\theta_n \quad \forall (z, w) \in P \times Q, \forall n \geq n_0.
\end{aligned}$$

Since $(\mathsf{s}_n(\tau), \mathsf{u}_n(\tau))$ weakly converges to $(\mathsf{s}(\tau), \mathsf{u}(\tau)) \neq (s_0(\tau), u_0(\tau))$ and $(\mathsf{s}_n(\tau), \mathsf{u}_n(\tau)) = (s_0(\tau), u_0(\tau)) + \delta_n(\tau)[(s_n(\tau), u_n(\tau)) - (s_0(\tau), u_0(\tau))]$, it has

$$\begin{aligned}
&\int_K \Big[(z(\tau) - \mathsf{s}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - \mathsf{u}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&\qquad +D_\gamma(z(\tau) - \mathsf{s}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \\
&= \lim_{n\to\infty} \int_K \Big[(z(\tau) - \mathsf{s}_n(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - \mathsf{u}_n(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau)) \\
&\qquad +D_\gamma(z(\tau) - \mathsf{s}_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \\
&\geq -\lim_{n\to\infty} \theta_n = 0, \quad \forall (z, w) \in P \times Q.
\end{aligned}$$

By Lemma 1, we obtain

$$\int_K \Big[(z(\tau) - \mathsf{s}(\tau))\frac{\partial f}{\partial s}(\pi_{\mathsf{s},\mathsf{u}}(\tau)) + (w(\tau) - \mathsf{u}(\tau))\frac{\partial f}{\partial u}(\pi_{\mathsf{s},\mathsf{u}}(\tau))$$
$$+ D_\gamma(z(\tau) - \mathsf{s}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{\mathsf{s},\mathsf{u}}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w) \in P \times Q. \tag{8}$$

This involves $(\mathsf{s}(\tau), \mathsf{u}(\tau)) \neq (s_0(\tau), u_0(\tau))$ is a solution of (CVI), which is a contradiction with the uniqueness of (CVI). Thus, $\{(s_n(\tau), u_n(\tau))\}$ is a bounded sequence having a convergent subsequence $\{(s_{n_k}(\tau), u_{n_k}(\tau))\}$ which converges to $(\bar{s}, \bar{u} \in P \times Q$ as $k \to \infty$. Next, for $(s_{n_k}, u_{n_k}), (z,w) \in P \times Q$, we obtain (see (7))

$$\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{s_{n_k},u_{n_k}}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{s_{n_k},u_{n_k}}(\tau))$$
$$+D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_{n_k},u_{n_k}}(\tau))\Big]d\tau$$
$$\leq \int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+ D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau. \tag{9}$$

Also, by (5), we obtain

$$\lim_{k\to\infty}\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{s_{n_k},u_{n_k}}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{s_{n_k},u_{n_k}}(\tau))$$
$$+ D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_{n_k},u_{n_k}}(\tau))\Big]d\tau \geq 0. \tag{10}$$

By (9) and (10), we get

$$\lim_{k\to\infty}\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0,$$
$$\Rightarrow \int_K \Big[(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - \bar{u}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$
$$+D_\gamma(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0.$$

By considering Lemma 1, it follows

$$\int_K \Big[(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s}(\pi_{\bar{s},\bar{u}}(\tau)) + (w(\tau) - \bar{u}(\tau))\frac{\partial f}{\partial u}(\pi_{\bar{s},\bar{u}}(\tau))$$
$$+D_\gamma(z(\tau) - \bar{s}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{\bar{s},\bar{u}}(\tau))\Big]d\tau \geq 0,$$

which shows that $(\bar{s}(\tau), \bar{u}(\tau))$ is a solution of (CVI). Hence, $(s_{n_k}(\tau), u_{n_k}(\tau)) \to (\bar{s}(\tau), \bar{u}(\tau))$, that is, $(s_{n_k}(\tau), u_{n_k}(\tau)) \to (s_0(\tau), u_0(\tau))$, involving $(s_n(\tau), u_n(\tau)) \to (s_0(\tau), u_0(\tau))$. □

Theorem 3. *Let* $\int_K f(\pi_{s,u}(\tau))d\tau$ *be hemicontinuous and monotone on the convex, compact and nonempty set* $P \times Q$*. Then (CVI) is generalized well-posed if and only if* $\mathcal{S}$ *is non-empty.*

Proof. Consider that (CVI) is generalized well-posed. Therefore, $\mathcal{S}$ is non-empty. Now, conversely, consider $\{(s_n(\tau), u_n(\tau))\}$ is an approximating sequence for (CVI). Then, there exists a sequence of positive real numbers $\theta_n \to 0$ satisfying

$$\int_K \Big[(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau)) + (w(\tau) - u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau))$$

$$+ D_\gamma(z(\tau) - s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau + \theta_n \geq 0, \quad \forall (z,w) \in P \times Q. \tag{11}$$

By hypothesis, $P \times Q$ is a compact set and, therefore, $\{(s_n(\tau), u_n(\tau))\}$ has a subsequence $\{(s_{n_k}(\tau), u_{n_k}(\tau))\}$ which converges to some pair $(s_0, u_0) \in P \times Q$. Since the integral functional $\int_K f(\pi_{s,u}(\tau))d\tau$ is monotone on $P \times Q$, for $(s_{n_k}, u_{n_k}), (z,w) \in P \times Q$, we have

$$\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{s_{n_k},u_{n_k}}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{s_{n_k},u_{n_k}}(\tau))$$

$$+D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_{n_k},u_{n_k}}(\tau))\Big]d\tau$$

$$\leq \int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$

$$+D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau.$$

By considering limit $k \to \infty$ in the above inequality, it implies

$$\lim_{k\to\infty}\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{s_{n_k},u_{n_k}}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{s_{n_k},u_{n_k}}(\tau))$$

$$+D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_{n_k},u_{n_k}}(\tau))\Big]d\tau$$

$$\leq \lim_{k\to\infty}\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$

$$+ D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau. \tag{12}$$

Since $\{(s_{n_k}(\tau), u_{n_k}(\tau))\}$ is an approximating subsequence in $P \times Q$, by (11), it follows

$$\lim_{k\to\infty}\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{s_{n_k},u_{n_k}}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{s_{n_k},u_{n_k}}(\tau))$$

$$+ D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_{n_k},u_{n_k}}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w) \in P \times Q. \tag{13}$$

By (12) and (13), we obtain

$$\lim_{k\to\infty}\int_K \Big[(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_{n_k}(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$

$$+D_\gamma(z(\tau) - s_{n_k}(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w) \in P \times Q,$$

$$\Rightarrow \int_K \Big[(z(\tau) - s_0(\tau))\frac{\partial f}{\partial s}(\pi_{z,w}(\tau)) + (w(\tau) - u_0(\tau))\frac{\partial f}{\partial u}(\pi_{z,w}(\tau))$$

$$+D_\gamma(z(\tau) - s_0(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{z,w}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w) \in P \times Q.$$

By Lemma 1, we get

$$\int_K \Big[(z(\tau)-s_0(\tau))\frac{\partial f}{\partial s}(\pi_{s_0,u_0}(\tau))+(w(\tau)-u_0(\tau))\frac{\partial f}{\partial u}(\pi_{s_0,u_0}(\tau))$$

$$+D_\gamma(z(\tau)-s_0(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_0,u_0}(\tau))\Big]d\tau \geq 0, \quad \forall (z,w)\in P\times Q,$$

which implies that $(s_0(\tau),u_0(\tau))\in\mathcal{S}$. □

Theorem 4. *Let* $\int_K f(\pi_{s,u}(\tau))d\tau$ *be hemicontinuous and monotone on the convex, compact and nonempty set* $P\times Q$*. Then (CVI) is generalized well-posed if there exists* $\theta>0$ *such that* $\mathcal{S}_\theta\neq\emptyset$ *and it is bounded.*

Proof. Consider $\theta>0$ with $\mathcal{S}_\theta$ is nonempty and bounded, and $\{(s_n(\tau),u_n(\tau))\}$ is an approximating sequence for (CVI). Thus, there exists a sequence of positive real numbers $\theta_n\to 0$ satisfying

$$\int_K \Big[(z(\tau)-s_n(\tau))\frac{\partial f}{\partial s}(\pi_{s_n,u_n}(\tau))+(w(\tau)-u_n(\tau))\frac{\partial f}{\partial u}(\pi_{s_n,u_n}(\tau))$$

$$+D_\gamma(z(\tau)-s_n(\tau))\frac{\partial f}{\partial s_\gamma}(\pi_{s_n,u_n}(\tau))\Big]d\tau+\theta_n \geq 0, \quad \forall (z,w)\in P\times Q,$$

which involves $(s_n(\tau),u_n(\tau))\in\mathcal{S}_\theta$, $\forall n>m$ (see m as a positive integer). We get $\{(s_n(\tau),u_n(\tau))\}$ is a bounded sequence with a convergent subsequence $\{(s_{n_k}(\tau),u_{n_k}(\tau))\}$ which weakly converges to $(s_0(\tau),u_0(\tau))$ as $k\to\infty$. In the same manner of the proof of Theorem 3, we obtain $(s_0(\tau),u_0(\tau))\in\mathcal{S}$ and the proof is complete. □

Next, to highlight the theoretical elements derived in the paper, a real-life application is presented to which this approach applies and for which the previous methods do not work.

Illustrative application. Let $m=2$, $K=[0,1]^2=[0,1]\times[0,1]$ and $P\times Q=C^1(K,[-10,10])\times C(K,[-10,10])$. For $f(\pi_{s,u}(\tau))=u^2(\tau)+e^{s(\tau)}-s(\tau)$, let us extremize the mass of the flat plate K, having a controlled density given by $\frac{\partial f}{\partial s}(\pi_{s,u})(z-s)+\frac{\partial f}{\partial s_\gamma}(\pi_{s,u})D_\gamma(z-s)+\frac{\partial f}{\partial u}(\pi_{s,u})(w-u)$, for any $(z,w)\in P\times Q$, that depends on the current point, such that the following controlled dynamical system $s_\gamma(\tau)=u(\tau)$, $\forall\tau\in K$, together with the boundary conditions $(s,u)|_{\partial K}=0$, are satisfied.

To solve the above concrete mechanical-physics problem, we consider

$$f(\pi_{s,u}(\tau))=u^2(\tau)+e^{s(\tau)}-s(\tau)$$

and the controlled variational inequality (CVI-1): *Find* $(s,u)\in P\times Q$ *such that*

$$\int_K \Big[2(w(\tau)-u(\tau))u(\tau)+(z(\tau)-s(\tau))(e^{s(\tau)}-1)\Big]d\tau^1 d\tau^2\geq 0,\ \forall(z,w)\in P\times Q,$$

$$(s,u)|_{\partial K}=0, \quad s_\gamma=u.$$

We have $\mathcal{S}=\{(0,0)\}$ and the functional $\int_K f(\pi_{s,u}(\tau))d\tau$ is hemicontinuous and monotone on the closed, convex and nonempty set $P\times Q=C^1(K,[-10,10])\times C(K,[-10,10])$. All the hypotheses of Theorem 2 are satisfied. Therefore, we obtain the controlled variational inequality (CVI-1) is well-posed. Also, $\mathcal{S}_\theta=\{(0,0)\}$ and consequently, $\mathcal{S}_\theta\neq\emptyset$ and diam $\mathcal{S}_\theta\to 0$ as $\theta\to 0$. By using Theorem 1, we obtain the controlled variational inequality problem (CVI-1) is well-posed.

4. Conclusions

In this paper, well-posedness and generalized well-posedness have been analyzed for a class of commanded variational inequalities by introducing the new variants for hemicontinuity, monotonicity, and pseudomonotonicity associated with the considered functional. More concretely, under suitable hypotheses, we have established that the well-posedness can be analyzed in terms of the existence and uniqueness of the solution. Also, sufficient conditions have been formulated and proved for the generalized well-posedness by assuming the boundedness of approximating solution set. In addition, some examples have been presented to illustrate the theoretical results.

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