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A Stochastic Semi-Parametric SEIR Model with Infectivity in an Incubation Period

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Abstract: This paper introduces stochastic disturbances into a semi-parametric SEIR model with infectivity in an incubation period. The model combines the randomness of disease transmission and the nonlinearity of transmission rate, providing a flexible framework for more accurate description of the process of infectious disease transmission. On the basis of the discussion of the deterministic model, the stochastic semi-parametric SEIR model is studied. Firstly, we use Lyapunov analysis to prove the existence and uniqueness of global positive solutions for the model. Secondly, the conditions for disease extinction are established, and appropriate stochastic Lyapunov functions are constructed to discuss the asymptotic behavior of the model's solution at the disease-free equilibrium point of the deterministic model. Finally, the specific transmission functions are enumerated, and the accuracy of the results are demonstrated through numerical simulations.

Keywords: stochastic SEIR model; semi-parametric; nonlinear incidence rate; stability; disease-free equilibrium

MSC: 34F05; 60H10; 92B05

1. Introduction

Infectious diseases once threatened people's lives and health, and the transmission patterns of various infectious diseases are relatively complex. Not only are the channels of disease transmission different, but some diseases do not infect again after infection, and some diseases have the ability to transmit before they have symptoms. People summarize the transmission patterns of diseases through the study of infectious diseases. Based on this, strategies and measures are formulated to control and eradicate diseases. Mathematical modeling plays a crucial role in epidemiology by describing disease transmission mechanisms and studying disease behavior. One classical infectious disease model is the SIR model, categorizing the population into susceptible, infected, and removed compartments, represented by $S(t)$, $I(t)$ and $R(t)$, respectively, at time t. However, many diseases do not erupt immediately, and have a latent period. Hence, SEIR models with latent periods [1–5] have been widely studied. Scholars usually approach improving models from a more practical perspective. For instance, considering that the transmission rate cannot remain constant over time, some nonlinear transmission rate models [6–12] have been developed. For example, Sun. et al. [13] investigated an SEIR epidemic model featuring a nonlinear transmission rate αIS^q, constant recovery rate, and disease-induced mortality rate. It was shown that the global dynamics are completely determined by the contact number R_0. If $R_0 \leq 1$, the disease-free equilibrium is globally stable and the disease dies out. If $R_0 > 1$, the unique endemic equilibrium is globally stable. Naim. et al. [14] discussed the global existence and uniqueness, non-negativity, and finiteness of solutions pertaining to an SEIR model characterized by a nonlinear incidence rate given by the function $f(S, I)I + g(S, E)E$. The study demonstrated that the model features two primary equilibria: the disease-free equilibrium and the endemic equilibrium. Utilizing Lyapunov functionals in conjunction

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with LaSalle's invariance principle, the authors have established the global asymptotic stability of these equilibria. Jiao et al. [15] established an SEIR model with latent infectious and susceptible populations for home isolation. The specific model is as follows:

$$\begin{cases} dS(t) = \{A - \mu S(t) - \beta(1-\theta_1)S(t)[I(t) + \varepsilon_1 E(t)]\}dt, \\ dE(t) = \{\beta(1-\theta_1)S(t)[I(t) + \varepsilon_1 E(t)] - (\mu + k)E(t)\}dt, \\ dI(t) = [kE(t) - (\mu + p + \gamma)I(t)]dt, \\ dR(t) = [(p + \varepsilon_2\gamma)I(t) - \mu R(t)]dt. \end{cases} \tag{1}$$

where A denotes the population input rate; β denotes the infection rate from susceptible S to exposed E; θ_1 denotes the home isolation rate of susceptible populations; ε_1 denotes the infection rate during the latent period, and $0 < \varepsilon_1 \leq 1$; ε_2 denotes the recovery rate from the infected I compartment; μ denotes the natural death rate; γ denotes the recovery rate from the disease; $\frac{1}{k}$ denotes the average duration of the latent period; and p denotes the disease-induced mortality rate. For this model, the authors defined the basic reproductive number R_0 and proved that when $R_0 \leq 1$, the disease-free equilibrium point is asymptotically stable, and when $R_0 > 1$, the endemic equilibrium point is asymptotically stable.

With the advancement of epidemiology, people are considering a more comprehensive range of factors in infectious disease models. For example, quarantine measures, regional culture, or the impact of vaccination on transmission. Gourieroux and Lu [16] combined the classic SIR model with semi-parametric properties allows for flexible handling of many data that are not known to follow a distribution or are not of a parametric distribution type. This enables the factors affecting the spread of the disease mentioned above to be captured by a transmission function $a(S(t))$, they established the following semi-parametric SIR model:

$$\begin{cases} dS(t) = a(S(t))I(t)dt, \\ dI(t) = [a(S(t))I(t) - \gamma I(t)]dt, \\ dR(t) = \gamma I(t)dt. \end{cases} \tag{2}$$

The exact solution of this model was derived, and infinite dimensional uncertainty was introduced to increase the flexibility of the model. In fact, disease transmission is inevitably influenced by environmental perturbations. Therefore, many stochastic models [17–20] of infectious diseases have been developed and have rapidly evolved since the beginning of the 21st century. Li and Guo [21] developed a stochastic SEIR epidemic model with standard incidence and vertical transmission. The primary goal of their study is to determine whether stochastic environmental disturbances affect dynamic features of the epidemic model. The existence, uniqueness, and boundedness of global positive solutions was stated. Li [22] proposed a stochastic SIR model with general nonlinear incidence and Lévy jumps. He has introduced the Lévy jump process to capture sudden and large-scale random processes such as earthquakes, volcanic eruptions, and tsunamis, which has certain practical significance. But research on stochastic semi-parameter SEIR models is currently limited. Based on this, this paper proposes a class of stochastic semi-parametric SEIR models with latent infectivity, which take the following form:

$$\begin{cases} dS(t) = \{A - \mu S(t) - a(S(t))[I(t) + \varepsilon_1 E(t)]\}dt + \sigma_1 S(t)dB_1(t), \\ dE(t) = \{a(S(t))[I(t) + \varepsilon_1 E(t)] - (\mu + k)E(t)\}dt + \sigma_2 E(t)dB_2(t), \\ dI(t) = \{kE(t) - (\mu + p + \gamma)I(t)\}dt + \sigma_3 I(t)dB_3(t), \\ dR(t) = [(p + \varepsilon_2\gamma)I(t) - \mu R(t)]dt + \sigma_4 R(t)dB_4(t). \end{cases} \tag{3}$$

where the function $a(S(t))$ represents the instantaneous transmission rate of the disease. Assuming that the mortality rate is affected by stochastic disturbance such as environmental climate change and population mobility, and is replaced by white noise, the parameter μ is replaced by $\mu + \sigma_i dB_i(t)$ $(i = 1, 2, 3, 4)$, where $B_i(t)$ $(i = 1, 2, 3, 4)$ are independent standard Brownian motions, σ_i $(i = 1, 2, 3, 4)$ are non-negative random noise intensities used to describe the volatility of disturbances.

The organization consists of the following: In Section 2, we discuss the basic reproductive number and stability of equilibrium points of the deterministic SEIR model corresponding to the stochastic model (3). Based on this, we prove the existence and uniqueness of the global positive solutions for the stochastic model (3). Section 3 derives the conditions for disease extinction in the stochastic model (3). In Section 4, we discuss the sufficient conditions for the asymptotic behavior of the solutions of the stochastic model (3) around its corresponding deterministic model's disease-free equilibrium point. Finally, the reliability of the results is demonstrated through numerical simulations.

2. The Existence and Uniqueness of the Global Solution

In this section, we first analyze the basic reproductive number, the disease-free equilibrium point, and stability of the deterministic semi-parameter SEIR model corresponding to model (3). Finally, based on this analysis, we discuss the existence and uniqueness of the global solutions for the stochastic model (3).

Lemma 1. *The deterministic semi-parameter SEIR model corresponding to model (3) is*

$$\begin{cases} dS(t) = \{A - \mu S(t) - a(S(t))[I(t) + \varepsilon_1 E(t)]\}dt, \\ dE(t) = \{a(S(t))[I(t) + \varepsilon_1 E(t)] - (\mu + k)E(t)\}dt, \\ dI(t) = \{kE(t) - (\mu + p + \gamma)I(t)\}dt, \\ dR(t) = [(p + \varepsilon_2\gamma)I(t) - \mu R(t)]dt. \end{cases} \tag{4}$$

The disease-free equilibrium point is $P_1 := (\frac{A}{\mu}, 0, 0, 0)$*, and the basic reproductive number is*

$$R_0 := \frac{\varepsilon a(S^0)}{(\mu + k)} + \frac{ka(S^0)}{(\mu + k)(\mu + p + \gamma)},$$

where $S^0 := \frac{A}{\mu}$.

Proof. Applying the next-generation matrix method [23] to compute the basic reproductive number, let

$$Q := \begin{bmatrix} E(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} a(S(t))[I(t) + \varepsilon_1 E(t)] \\ 0 \end{bmatrix} - \begin{bmatrix} (\mu + k)E(t) \\ -kE(t) + (\mu + p + \gamma)I(t) \end{bmatrix},$$

and

$$F(t) := \begin{bmatrix} a(S(t))[I(t) + \varepsilon_1 E(t)] \\ 0 \end{bmatrix}, V(t) := \begin{bmatrix} (\mu + k)E(t) \\ -kE(t) + (\mu + p + \gamma)I(t) \end{bmatrix}.$$

Clearly, $P_1 = (\frac{A}{\mu}, 0, 0, 0)$ is the disease-free equilibrium point of model (4), and the Jacobian matrices of $F(t)$, $V(t)$ and at P_1 are

$$F_1 := \left.\frac{\partial F}{\partial t}\right|_{P_1} = \begin{bmatrix} \varepsilon_1 a(S^0) & a(S^0) \\ 0 & 0 \end{bmatrix}, V_1 := \left.\frac{\partial V}{\partial t}\right|_{P_1} = \begin{bmatrix} \mu + k & 0 \\ -k & \mu + p + \gamma \end{bmatrix}.$$

Hence,

$$F_1 V_1^{-1} = \begin{bmatrix} \frac{\varepsilon_1 a(S^0)}{(\mu+k)} + \frac{ka(S^0)}{(\mu+k)(\mu+p+\gamma)} & \frac{(\mu+k)a(S^0)}{(\mu+k)(\mu+p+\gamma)} \\ 0 & 0 \end{bmatrix}.$$

Therefore, the spectral radius of $F_1 {V_1}^{-1}$ is $\frac{\varepsilon a[S^0]}{(\mu+k)} + \frac{ka[S^0]}{(\mu+k)(\mu+p+\gamma)}$, which represents the basic reproductive number. □

Assumption 1.

(1) *The instantaneous transmission rate of the disease satisfies* $a(S(0)) = 0$, $a'(S(t)) > 0$, *meaning the more susceptible people there are, the greater the probability of getting infected and spreading the disease;*
(2) *Suppose* $a(S)$ *is bounded, meaning there exists* M, *such that* $a(S) \leq M$;
(3) *Suppose* $a'(S) \leq a(S)$.

Lemma 2. *When* $R_0 \leq 1$, *the disease-free equilibrium point* P_1 *of model* (4) *is globally asymptotically stable.*

Proof. Let $V_1 = kE + (\mu + k)I$; then, we have

$$\frac{dV_1}{dt} = ka(S)(I + \varepsilon_1 E) - (\mu + k)(\mu + p + \gamma)I.$$

Since $S \leq S^0$ and $a'(S) > 0$, we have $a(S) \leq a(S^0)$. Therefore, when $R_0 \leq 1$, we further have

$$\begin{aligned}
\frac{dV_1}{dt} &\leq ka\left(S^0\right)\left(I + \varepsilon_1 \frac{\mu + p + \gamma}{k} I\right) - (\mu + k)(\mu + p + \gamma)I \\
&\leq (\mu + k)(\mu + p + \gamma)I\left\{\frac{ka(S^0)}{(\mu + k)(\mu + p + \gamma)} + \frac{\varepsilon_1 a(S^0)}{\mu + k} - 1\right\} \\
&\leq (\mu + k)(\mu + p + \gamma)I(R_0 - 1) \\
&\leq 0.
\end{aligned}$$

According to LaSalle's asymptotic stability theorem [24], the disease-free equilibrium point is globally asymptotically stable. □

Then, we will use the Lyapunov function analysis method [25] to prove the existence and uniqueness of the stochastic semi-parameter SEIR model (3).

Theorem 1. *For any initial values* $(S(0), E(0), I(0), R(0)) \in R_+^4$, *for almost all* $t \geq 0$, *the model* (3) *has a unique positive solution* $(S(t), E(t), I(t), R(t))$, *and this solution remains in* R_+^4 *with probability 1.*

Proof. Since the coefficients of the model (3) satisfy the local Lipschitz condition, for any $(S(0), E(0), I(0), R(0)) \in R_+^4$, there exists a unique local positive solution $(S(t), E(t), I(t), R(t))$ $(t \in [0, \tau_e))$, where τ_e is the explosion time. If $\tau_e = +\infty$, then this indicates that the solutions of model (3) are global.

First of all, we prove that $(S(t), E(t), I(t), R(t))$ do not experience explosion at a finite time. Let $i_0 \geq 1$ be large enough that $(S(0), E(0), I(0), R(0))$ is located in the interval $\left[\frac{1}{i_0}, i_0\right]$. For each integer $i \geq i_0$, define a stopping time,

$$\tau_i := \inf\{t \in [0, \tau_e) : \min\{S, E, I, R\} \leq \frac{1}{i} \text{ or } \max\{S, E, I, R\} \geq i\}.$$

It is evident that τ_i increases as $i \to +\infty$. Let $\tau_\infty = \lim_{i \to \infty} \tau_i$, thus $\tau_\infty \leq \tau_e$ a.s.. If we can prove that $\tau_\infty = +\infty$ a.s., then we can conclude $\tau_e = +\infty$ a.s., and $(S(t), E(t), I(t), R(t)) \in R_+^4$ a.s.. If this statement is false, then there is a pair of constants $T \geq 0$ and $\varepsilon \in (0, 1)$, such that $P\{\tau_\infty \leq T\} > \varepsilon$, thus there exists an integer $i_1 \geq i_0$, such that:

$$P\{\tau_\infty \leq T\} > \varepsilon, \forall i \geq i_1.$$

Define a C^2 function V: $R_+^4 \to R_+$,

$$V(S,E,I,R) := (S - m - m\ln\frac{S}{m}) + (E - 1 - \ln E) + (I - 1 - \ln I) + (R - 1 - \ln R).$$

where m is an undetermined coefficient; then, using the *Itô* formula [26], we obtain

$$dV(S,E,I,R) = LV(S,E,I,R)dt + \sigma_1(S-m)dB_1 + \sigma_2(E-1)dB_2 + \sigma_3(I-1)dB_3 + \sigma_4(R-1)dB_4,$$

where

$$\begin{aligned}
LV(S,E,I,R) &= (1-\frac{m}{S})[A - \mu S - a(S)(I+\varepsilon_1 E)] + \frac{m}{2}\frac{1}{S^2}\sigma_1^2 S^2 \\
&\quad + (1-\frac{1}{E})[a(S)(I+\varepsilon_1 E) - (\mu+k)E] + \frac{1}{2}\frac{1}{E^2}\sigma_2^2 E^2 \\
&\quad + (1-\frac{1}{I})[kE - (\mu+p+\gamma)I] + \frac{1}{2}\frac{1}{I^2}\sigma_3^2 I^2 \\
&\quad + (1-\frac{1}{R})[(p+\varepsilon_2\gamma)I - \mu R] + \frac{1}{2}\frac{1}{R^2}\sigma_4^2 R^2 \\
&= A + m\mu + 3\mu + k + p + \gamma - \mu S - \mu E - \mu R - (\mu + \gamma - \varepsilon_2\gamma)I - \frac{mA}{S} \\
&\quad + \frac{ma(S)(I+\varepsilon_1 E)}{S} - \frac{a(S)(I+\varepsilon_1 E)}{E} - \frac{kE}{I} - \frac{(p+\varepsilon_2\gamma)I}{R} \\
&\quad + \frac{m\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\le A + m\mu + 3\mu + k + p + \gamma + \frac{m\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
&\quad + mM(I+\varepsilon_1 E) - \mu(I+\varepsilon_1 E).
\end{aligned}$$

Taking $m = \frac{\mu}{M}$, then we obtain

$$LV(S,E,I,R) \le A + m\mu + 3\mu + k + p + \gamma + \frac{m\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} := M_1.$$

Therefore, we have

$$dV(S,E,I,R) \le M_1 dt + \sigma_1(S-m)dB_1 + \sigma_2(E-1)dB_2 + \sigma_3(I-1)dB_3 + \sigma_4(R-1)dB_4. \quad (5)$$

Let $V(T\wedge\tau_i) := V(S(T\wedge\tau_i), E(T\wedge\tau_i), I(T\wedge\tau_i), R(T\wedge\tau_i))$, integrate Equation (5) from 0 to $T\wedge\tau_i$, and then take the expected value to obtain

$$\begin{aligned}
EV(T\wedge\tau_i) &\le V(S(0),E(0),I(0),R(0)) + M_1 E(T\wedge\tau_i) \\
&\le V(S(0),E(0),I(0),R(0)) + M_1 T.
\end{aligned}$$

Let $\Omega_i := \{\tau_i \le T\}$, for $i \ge i_1$; if $P(\Omega_i) \ge \varepsilon$, then we further have

$$EV(T\wedge\tau_i) = E[1_{\Omega_i} V(T\wedge\tau_i)] + E[1_{\Omega_i^c} V(T\wedge\tau_i)] \ge E[1_{\Omega_i} V(T\wedge\tau_i)].$$

where 1_{Ω_i} is the indicator function of Ω_i, for any $\omega \in \Omega_i$, $S(\tau_i,\omega)$, $E(\tau_i,\omega)$, $I(\tau_i,\omega)$ and $R(\tau_i,\omega)$ where at least one of them equals i or $\frac{1}{i}$. Therefore, we have

$$\begin{aligned}
& V(S(T\wedge\tau_i), E(T\wedge\tau_i), I(T\wedge\tau_i), R(T\wedge\tau_i) \\
\ge\ & \min\left\{ i - m - m\ln\frac{i}{m}, \frac{1}{i} - m + m\ln im, i - 1 - \ln i, \frac{1}{i} - 1 + \ln i \right\} \\
=\ & A(i).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned} V(S(0),E(0),I(0),R(0)) + M_1 T &\geq E[1_{\Omega_i} V(S(T\wedge\tau_i),E(T\wedge\tau_i),I(T\wedge\tau_i),R(T\wedge\tau_i))] \\ &\geq A(i)\varepsilon. \end{aligned}$$

As $i \to \infty$, $V(S(0),E(0),I(0),R(0)) + M_1 T \geq \infty$, which results in a contradiction. Hence, $\tau_e = \infty$, *a.s.* □

3. The Extinction of Disease

We will discuss the conditions for disease extinction in the stochastic semi-parameter SEIR model (3) in this section.

Theorem 2. *For any initial values* $(S(0),E(0),I(0),R(0)) \in R_+^4$, *let* $((S(t),E(t),I(t),R(t))$ *be the solution of model* (3). *If*

$$R_1 = \frac{2kM(k+\mu)}{k^2\frac{\sigma_2^2}{2} \wedge (k+\mu)^2(\mu+p+\gamma+\frac{\sigma_3^2}{2})} < 1, \varepsilon_1 \leq \frac{k}{\mu+k},$$

then we can obtain that

$$\limsup_{t\to\infty} \frac{\ln[kE+(k+\mu)I]}{t} \leq \frac{k^2\frac{\sigma_2^2}{2} \wedge (k+\mu)^2(\mu+p+\gamma+\frac{\sigma_3^2}{2})}{2(k+\mu)^2}(R_1-1) < 0, a.s..$$

Proof. Let $V := kE + (k+\mu)I$; using the *Itô* formula, we can obtain

$$\begin{aligned} d\ln V &= \frac{1}{V}\{k\{[a(s)(I+\varepsilon_1 E) - (k+\mu)E]dt + \sigma_2 E(t)dB_2(t)\} + (k+\mu)\{[kE-(\mu+p+\gamma)I]dt\}\} \\ &\quad + \frac{k\sigma_2 E}{V}dB_2 + \frac{(k+\mu)\sigma_3 I}{V}dB_3 - \frac{1}{2}\frac{1}{V^2}\left[k^2\sigma_2^2E^2 + (k+\mu)^2\sigma_3^2I^2\right]dt \\ &= \frac{[ka(s)(I+\varepsilon_1 E) - k(k+\mu)E]}{V} + \frac{k(k+\mu)E - (k+\mu)(\mu+p+\gamma)I}{V} \\ &\quad - \frac{1}{2}\frac{k^2\sigma_2^2E^2}{V^2} - \frac{1}{2}\frac{(k+\mu)^2\sigma_3^2I^2}{V^2} + \frac{k\sigma_2 E}{V}dB_2 + \frac{(k+\mu)\sigma_3 I}{V}dB_3. \end{aligned}$$

$$\begin{aligned} d\ln V &= \left\{\frac{ka(s)(I+\varepsilon_1 E)}{kE+(k+\mu)I} - \frac{2(k+\mu)(\mu+p+\gamma)[kE+(k+\mu)I]I + k^2\sigma_2^2E^2 + (k+\mu)^2\sigma_3^2I^2}{2[kE+(k+\mu)I]^2}\right\}dt \\ &\quad + \frac{k\sigma_2 E}{V}dB_2 + \frac{(k+\mu)\sigma_3 I}{V}dB_3 \\ &\leq \left\{\frac{ka(s)}{(k+\mu)} - \frac{2(\mu+p+\gamma)(k+\mu)^2I^2 + k^2\sigma_2^2E^2 + (k+\mu)^2\sigma_3^2I^2}{2(k+\mu)^2(E+I)^2}\right\}dt \\ &\quad + \frac{k\sigma_2 E}{V}dB_2 + \frac{(k+\mu)\sigma_3 I}{V}dB_3 \\ &\leq \left\{\frac{kM}{k+\mu} - \frac{k^2\frac{\sigma_2^2}{2} \wedge (k+\mu)^2(\mu+p+\gamma+\frac{\sigma_3^2}{2})}{2(k+\mu)^2}\right\}dt + \frac{k\sigma_2 E}{V}dB_2 + \frac{(k+\mu)\sigma_3 I}{V}dB_3. \end{aligned} \tag{6}$$

Integrating Equation (6) from $0 \to t$ first, then dividing by t, we further have

$$\begin{aligned} \frac{\ln V(t) - \ln V(0)}{t} &\leq \frac{kM}{k+\mu} - \frac{1}{2}\frac{k^2\frac{\sigma_2^2}{2} \wedge (k+\mu)^2(\mu+p+\gamma+\frac{\sigma_3^2}{2})}{(k+\mu)^2} \\ &\quad + \frac{1}{t}\left(\int_o^t \frac{k\sigma_2 E}{V(s)}dB_2(s) + \int_o^t \frac{(k+\mu)\sigma_3 I}{V(s)}dB_3(s)\right). \end{aligned}$$

By taking the upper limit on both sides of the inequality and subsequently applying the strong law of large numbers, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln V(t)-\ln V(0)}{t} &\leq \frac{kM}{k+\mu}-\frac{1}{2}\frac{k^2\frac{\sigma_2^2}{2}\wedge(k+\mu)^2(\mu+p+\gamma+\frac{\sigma_3^2}{2})}{(k+\mu)^2}\\ &\leq \frac{k^2\frac{\sigma_2^2}{2}\wedge(k+\mu)^2(\mu+p+\gamma+\frac{\sigma_3^2}{2})}{2(k+\mu)^2}(R_1-1).\end{aligned}$$

From the given condition $R_1<1$, therefore, we further have

$$\limsup_{t\to\infty}\frac{\ln V(t)-\ln V(0)}{t}\leq 0.$$

It follows that $\lim_{t\to\infty}E(t)=0$, $\lim_{t\to\infty}I(t)=0$, *a.s.*. So, we obtain that $\lim_{t\to\infty}R(t)=0$, *a.s.* □

4. Stability of the Disease-Free Equilibrium

We will discuss the establishment of sufficient conditions for the average vibration of the model (3) around P_1 in this section.

Theorem 3. *For any initial values* $(S(0),E(0),I(0),R(0))\in R_+^4$, *let* $(S(t),E(t),I(t),R(t))$ *be the solution of model* (3). *If* $R_0\leq 1$, $\zeta_i>0(i=1,2,3,4)$, *then we can obtain*

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[\zeta_1(S(v)-\frac{A}{\mu})^2+\zeta_2E^2(v)+\zeta_3I^2(v)+\zeta_4R^2(v)\right]dv\\ \leq\frac{1}{2}\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2(\frac{A}{\mu})^2,\end{aligned}$$

where

$$\begin{aligned}\zeta_1 &= \mu-\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2,\\ \zeta_2 &= \left[\frac{(\mu+k)}{4}-\frac{1}{2}\right]\sigma_2^2,\\ \zeta_3 &= \frac{(\mu+k)(\mu+p+\gamma)}{4k^2}\left[\frac{(\mu+p+\gamma)}{2}-\sigma_3^2\right],\\ \zeta_4 &= \frac{\mu(\mu+k)(\mu+p+\gamma)^2(\mu-\sigma_4^2)}{8k^2(p+\varepsilon_2\gamma)^2}.\end{aligned}$$

Proof. Define a series of C^2-functions,

$$V_1:=\frac{1}{2}(S-\frac{A}{\mu})^2,\qquad V_2:=\frac{I^2}{2},\qquad V_3:=\frac{1}{2}R^2,$$

$$V_4:=\frac{1}{2}(S-\frac{A}{\mu}+E)^2,\qquad V_5:=E+\frac{\mu+k}{k+\varepsilon_1(\mu+p+\gamma)}I.$$

From the *Itô* formula, we can obtain

$$LV_1=-\mu(S-\frac{A}{\mu})^2-a(S)(I+\varepsilon E)(S-\frac{A}{\mu})+\frac{1}{2}\sigma_1^2S^2. \tag{7}$$

$$LV_2\leq\frac{1}{2}\frac{k^2}{\mu+p+\gamma}E^2-\frac{1}{2}(\mu+p+\gamma)I^2+\frac{1}{2}\sigma_3^2I^2. \tag{8}$$

$$LV_3 \leq -\frac{1}{2}\mu R^2 + \frac{(p+\varepsilon_2\gamma)^2 I^2}{2\mu} + \frac{1}{2}\sigma_4^2 R^2. \tag{9}$$

$$LV_4 \leq \left[\frac{(2\mu+k)^2}{2(\mu+k)} - \mu\right](S - \frac{A}{\mu})^2 - \frac{(\mu+k)}{2}E^2 + \frac{1}{2}\sigma_1^2 S^2 + \frac{1}{2}\sigma_2^2 E^2. \tag{10}$$

$$\begin{aligned} LV_5 &= a(S)(I+\varepsilon_1 E) - (\mu+k)E + \frac{\mu+k}{k+\varepsilon_1(\mu+p+\gamma)}(kE - (\mu+p+\gamma)I) \\ &= a(S)(I+\varepsilon_1 E) - \frac{\varepsilon_1(\mu+p+\gamma)(\mu+k)E}{k+\varepsilon_1(\mu+p+\gamma)} - \frac{(\mu+p+\gamma)I}{k+\varepsilon_1(\mu+p+\gamma)} \\ &= a(S)(I+\varepsilon_1 E) - \frac{(\mu+p+\gamma)(\mu+k)(I+\varepsilon_1 E)}{k+\varepsilon_1(\mu+p+\gamma)} \\ &= \left[a(S) - a(S^0)\right](I+\varepsilon_1 E) - \frac{(\mu+k)(\mu+p+\gamma)(I+\varepsilon E)}{k+\varepsilon_1(\mu+p+\gamma)}(1-R_0). \end{aligned}$$

From the mean value theorem, we have

$$LV_5 \leq a'(\hat{S})(S - \frac{A}{\mu})(I+\varepsilon E) - \frac{(\mu+k)(\mu+p+\gamma)(I+\varepsilon E)}{k+\varepsilon_1(\mu+p+\gamma)}(1-R_0). \tag{11}$$

Combining Equations (7) and (11), we obtain

$$\begin{aligned} LV_1 + LV_5 &\leq -\mu(S - \frac{A}{\mu})^2 - a(S)(I+\varepsilon E)(S - \frac{A}{\mu}) + \frac{1}{2}\sigma_1^2 S^2 \\ &\quad + a'(\hat{S})(S - \frac{A}{\mu})(I+\varepsilon E) - \frac{(\mu+k)(\mu+p+\gamma)(I+\varepsilon E)}{k+\varepsilon_1(\mu+p+\gamma)}(1-R_0) \\ &\leq -\mu(S - \frac{A}{\mu})^2 - (a(S) - a'(\hat{S}))(I+\varepsilon E)(S - \frac{A}{\mu}) + \frac{1}{2}\sigma_1^2 S^2 \\ &\quad - \frac{(\mu+k)(\mu+p+\gamma)(I+\varepsilon E)}{k+\varepsilon_1(\mu+p+\gamma)}(1-R_0). \end{aligned} \tag{12}$$

Combining Equations (8) and (9), we obtain

$$\begin{aligned} LV_2 + \frac{\mu(\mu+p+\gamma)}{2(p+\varepsilon_2\gamma)^2}LV_3 &\leq \frac{1}{2}\frac{k^2}{\mu+p+\gamma}E^2 - \frac{1}{2}(\mu+p+\gamma)I^2 + \frac{1}{2}\sigma_3^2 I^2 \\ &\quad + \frac{\mu(\mu+p+\gamma)}{2(p+\varepsilon_2\gamma)^2}\left[-\frac{\mu R}{2} + \frac{(p+\varepsilon_2\gamma)I^2}{2\mu} + \frac{\sigma_4^2 R^2}{2}\right] \\ &\leq \frac{1}{2}\frac{k^2 E^2}{\mu+p+\gamma} - \frac{(\mu+p+\gamma)I^2}{4} + \frac{1}{2}\sigma_3^2 I^2 \\ &\quad + \frac{\mu(\mu+p+\gamma)(\sigma_4^2 - \mu)R^2}{4(p+\varepsilon_2\gamma)^2}. \end{aligned} \tag{13}$$

Define a C^2-function,

$$V := \frac{(2\mu+k)^2}{2\mu(\mu+k)}(V_1+V_5) + \frac{(\mu+k)(\mu+p+\gamma)}{2k^2}\left[V_2 + \frac{\mu(\mu+p+\gamma)}{2(p+\varepsilon_2\gamma)^2}V_3\right] + V_4.$$

From the *Itô* formula, and by substituting into (10), (12) and (13), we have

$$
\begin{aligned}
LV &= \frac{(2\mu+k)^2}{2\mu(\mu+k)}\Bigg\{-\mu(S-\frac{A}{\mu})^2-\big[a(S)-a'(\hat{S})\big](I+\varepsilon E)(S-\frac{A}{\mu})\\
&\quad -\frac{(\mu+k)(\mu+p+\gamma)I}{k}(1-R_0)+\frac{1}{2}\sigma_1^2S^2\Bigg\}\\
&\quad +\frac{(\mu+k)(\mu+p+\gamma)}{2k^2}\left[\frac{1}{2}\frac{k^2E^2}{\mu+p+\gamma}-\frac{(\mu+p+\gamma)I^2}{4}+\frac{1}{2}\sigma_3^2I^2+\frac{\mu(\mu+p+\gamma)(\sigma_4^2-\mu)R^2}{4(p+\varepsilon_2\gamma)^2}\right]\\
&\quad +\left[\frac{(2\mu+k)^2}{2(\mu+k)}-\mu\right](S-\frac{A}{\mu})^2-\frac{(\mu+k)}{2}E^2+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}\sigma_2^2E^2\\
&\leq \frac{1}{2}\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2(\frac{A}{\mu})^2-\left\{\mu-\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2\right\}(S-\frac{A}{\mu})^2\\
&\quad -\left[\frac{(\mu+k)}{4}-\frac{1}{2}\right]\sigma_2^2E^2-\frac{(\mu+k)(\mu+p+\gamma)}{4k^2}\left[\frac{(\mu+p+\gamma)}{2}-\sigma_3^2\right]I^2\\
&\quad -\frac{\mu(\mu+k)(\mu+p+\gamma)^2(\mu-\sigma_4^2)}{8k^2(p+\varepsilon_2\gamma)^2}R^2.
\end{aligned}
\tag{14}
$$

First, we integrate Equation (14) from 0 to t, then simultaneously take expectations on both sides of the equation, then we obtain

$$
\begin{aligned}
EV(t)-EV(0) &\leq -\left\{\mu-\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2\right\}E\int_0^t(S(\mu)-\frac{A}{\mu})^2d\mu\\
&\quad -\left[\frac{(\mu+k)}{4}-\frac{1}{2}\right]\sigma_2^2E\int_0^tE(\mu)d\mu\\
&\quad -\frac{(\mu+k)(\mu+p+\gamma)}{4k^2}\left[\frac{(\mu+p+\gamma)}{2}-\sigma_3^2\right]E\int_0^tI(\mu)d\mu\\
&\quad -\frac{\mu(\mu+k)(\mu+p+\gamma)^2(\mu-\sigma_4^2)}{8k^2(p+\varepsilon_2\gamma)^2}E\int_0^tR(\mu)d\mu\\
&\quad +\frac{1}{2}\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2(\frac{A}{\mu})^2.
\end{aligned}
$$

Therefore, we further obtain

$$
\limsup_{t\to\infty}\frac{1}{t}E\int_0^1\zeta_1(S(v)-\frac{A}{\mu})^2+\zeta_2E^2(v)+\zeta_3I^2(v)+\zeta_4R^2(v)dv\leq\frac{1}{2}\left[\frac{(2\mu+k)^2}{2\mu(\mu+k)}+1\right]\sigma_1^2(\frac{A}{\mu})^2.
$$

□

5. Simulation

In this section, we demonstrate the correctness of the obtained results by enumerating specific transmission functions and conducting numerical simulations using the Milstein high-order method [27].

Example 1. *We take $a[S(t)]=\beta S(t)$, and let $M=\beta\times\frac{A}{\mu}$.*

Applying the Milstein high-order method, the discrete equations of the model is as follows.

$$
\begin{cases}
S(n+1)=S(n)+\{A-\mu S(n)-\beta S(n)[I(n)+\varepsilon_1E(n)]\}\Delta t+\sigma_1S(n)\xi_{1n}\sqrt{\Delta t}+\frac{\sigma_1^2}{2}S(n)(\xi_{1n}^2-1)\Delta t,\\
E(n+1)=E(n)+\{\beta S(n)[I(n)+\varepsilon_1E(n)]-(\mu+k)E(n)\}\Delta t+\sigma_2E(n)\xi_{2n}\sqrt{\Delta t}+\frac{\sigma_2^2}{2}E(n)(\xi_{2n}^2-1)\Delta t,\\
I(n+1)=I(n)+[kE(n)-(\mu+p+\gamma)I(n)]\Delta t+\sigma_3I(n)\xi_{3n}\sqrt{\Delta t}+\frac{\sigma_3^2}{2}I(n)(\xi_{3n}^2-1)\Delta t,\\
R(n+1)=R(n)+[(p+\varepsilon_2\gamma)I(n)-\mu R(n)]\Delta t+\sigma_4R(n)\xi_{4n}\sqrt{\Delta t}+\frac{\sigma_4^2}{2}R(n)(\xi_{4n}^2-1)\Delta t.
\end{cases}
$$

where $\xi_{mn}(m = 1, 2, 3, 4, n = 1, 2, \cdots, n_1)$ *are random variables following the standard normal distribution* $N(0, 1)$*, such that* $\Delta B_n = \xi_{mn}\sqrt{\Delta t}$*. Assume initial values* $S(0) = 100, E(0) = 30, I(0) = 40, R(0) = 30$*, and parameters* $A = 10$, $\mu = 0.25$, $\beta = 0.002$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.3$, $k = 0.3$, $p = 0.2$, $\gamma = 0.2$*. Set* $n_1 = 20,000$*, and step size* $\Delta t = 0.005$*. The corresponding deterministic model can be simulated as shown in Figure 1.*

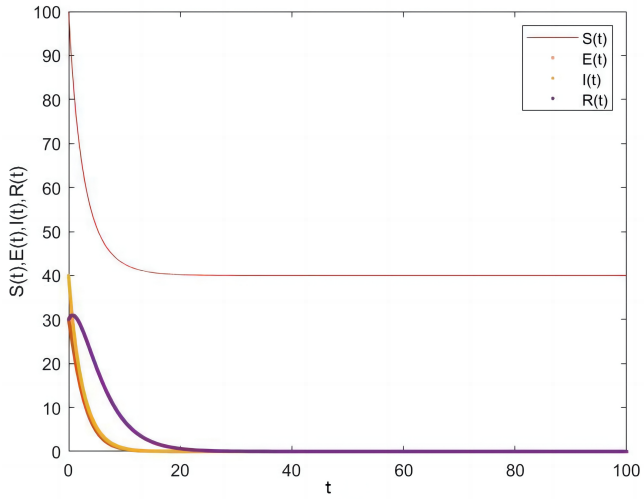

Figure 1. Deterministic model.

In the simulation processes depicted in Figure 2, take $\sigma_1 = 0.1$, $\sigma_2 = 0.8$, $\sigma_3 = 0.4$, $\sigma_4 = 0.1$*. We can observe that, due to the influence of noise disturbance, the solutions of the random model exhibit irregular up and down fluctuations compared to the deterministic model. And calculate* $R_1 = 0.9167$*, which satisfies the conditions of Theorem 2. It can be observed that the disease eventually dies out.*

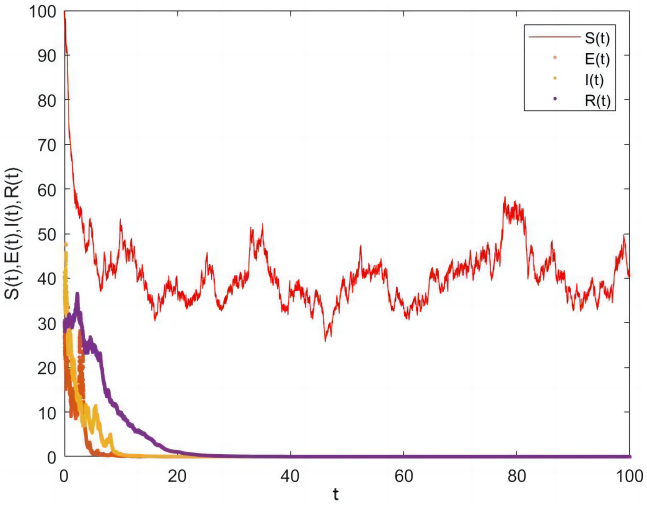

Figure 2. Stochastic model.

In the simulation processes depicted in Figures 3 and 4, take $\beta = 0.002$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\sigma_3 = 0.6$, $\sigma_4 = 0.5$*. The rest of the parameters are consistent with the above. Calculate* $R_0 = 0.8168$*, satisfying the conditions of Theorem 3. It can be observed that the solutions of the model fluctuate near the equilibrium point* $P_1 = \left(\frac{A}{\mu}, 0, 0, 0\right)$ *of the deterministic model.*

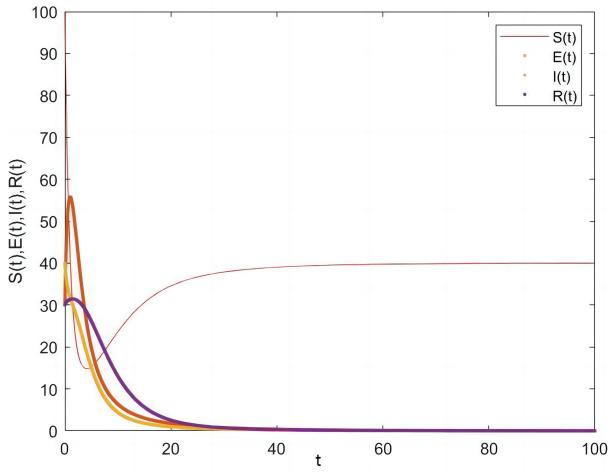

Figure 3. Deterministic model.

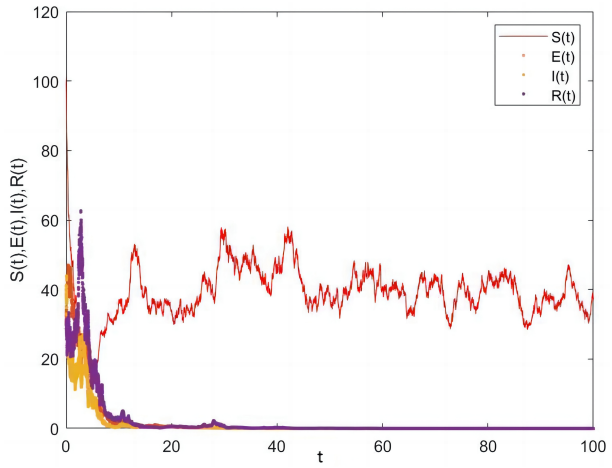

Figure 4. Stochastic model.

Example 2. *We take* $a[S(t)] = \beta S^2(t)$, *and let* $M = \beta \times \left(\frac{A}{\mu}\right)^2$.

Applying the Milstein high-order method, the discrete equations of the model is as follows.

$$\begin{cases} S(n+1) = S(n) + \{A - \mu S(n) - \beta S^2(n)[I(n) + \varepsilon_1 E(n)]\}\Delta t + \sigma_1 S(n)\xi_{1n}\sqrt{\Delta t} + \frac{\sigma_1^2}{2} S(n)(\xi_{1n}^2 - 1)\Delta t, \\ E(n+1) = E(n) + \{\beta S^2(n)[I(n) + \varepsilon_1 E(n)] - (\mu + k)E(n)\}\Delta t + \sigma_2 E(n)\xi_{2n}\sqrt{\Delta t} + \frac{\sigma_2^2}{2} E(n)(\xi_{2n}^2 - 1)\Delta t, \\ I(n+1) = I(n) + [kE(n) - (\mu + p + \gamma)I(n)]\Delta t + \sigma_3 I(n)\xi_{3n}\sqrt{\Delta t} + \frac{\sigma_3^2}{2} I(n)(\xi_{3n}^2 - 1)\Delta t, \\ R(n+1) = R(n) + [(p + \varepsilon_2\gamma)I(n) - \mu R(n)]\Delta t + \sigma_4 R(n)\xi_{4n}\sqrt{\Delta t} + \frac{\sigma_4^2}{2} R(n)(\xi_{4n}^2 - 1)\Delta\, t. \end{cases}$$

where $\xi_{mn}(m = 1,2,3,4, n = 1,2,\cdots,n_1)$ *are random variables following the standard normal distribution* $N(0,1)$, *such that* $\Delta B_n = \xi_{mn}\sqrt{\Delta t}$. *Assume initial values* $S(0) = 100, E(0) = 30$, $I(0) = 40, R(0) = 30$. *Set* $n_1 = 20{,}000$, *and step size* $\Delta t = 0.005$. *Take parameters* $A = 10$, $\mu = 0.4$, $\beta = 0.0001$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.3$, $k = 0.3$, $p = 0.2$, $\gamma = 0.2$. *The corresponding deterministic model can be simulated as shown in Figure 5.*

In the simulation processes depicted in Figure 6, set $\sigma_1 = 0.1$, $\sigma_2 = 0.8$, $\sigma_3 = 0.6$, $\sigma_4 = 0.5$. *Calculate* $R_1 = 0.9115$, *satisfying the conditions of Theorem 2. It can be observed that the disease eventually dies out.*

Set $A = 10$, $\mu = 0.3$, $\beta = 0.001$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.3$, $k = 0.3$, $p = 0.2$, $\gamma = 0.2$; *the corresponding deterministic model can be simulated as shown in Figure 7. In the simulation processes depicted in Figure 8, set* $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $\sigma_3 = 0.6$, $\sigma_4 = 0.5$. *The calculation yields* $R_0 = 0.8122$, *satisfying the conditions of Theorem 3. It can be observed that the solutions of the model fluctuate near the equilibrium point* $P_1 = \left(\frac{A}{\mu}, 0, 0, 0\right)$ *of the deterministic model.*

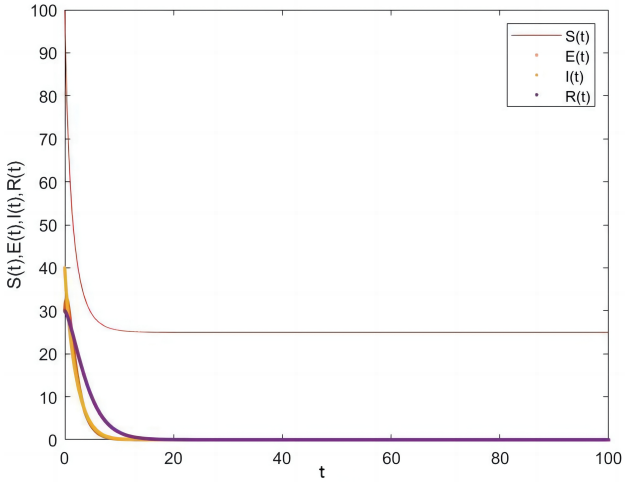

Figure 5. Deterministic model.

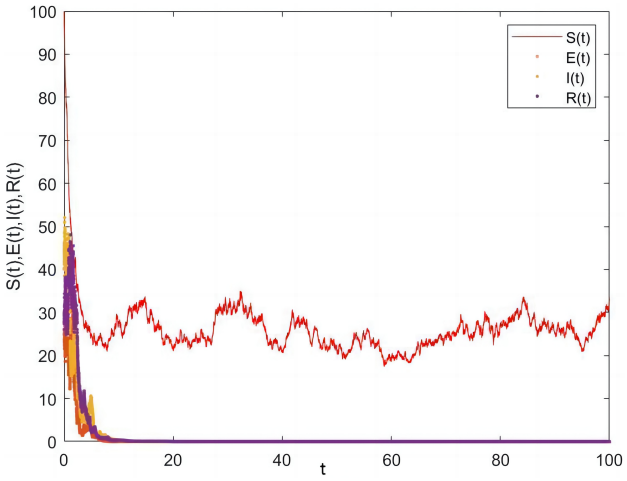

Figure 6. Stochastic model.

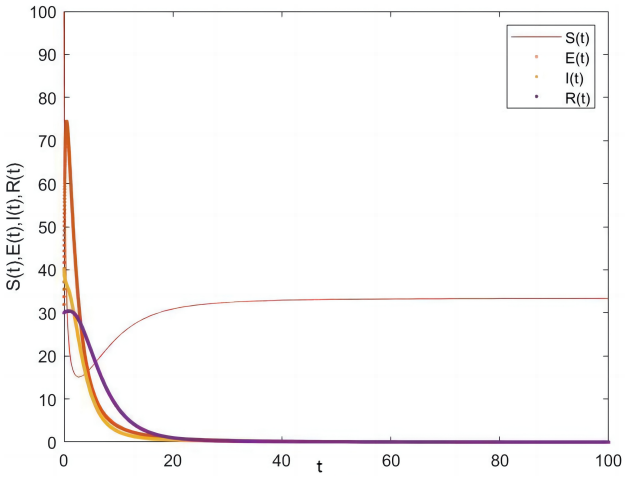

Figure 7. Deterministic model.

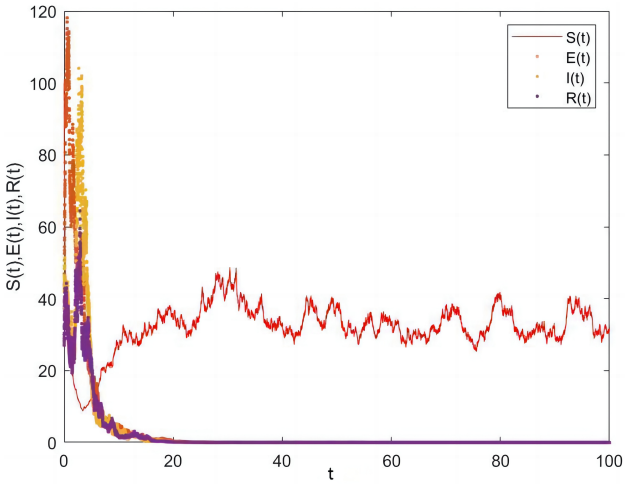

Figure 8. Stochastic model.

6. Conclusions

This paper discusses the global existence and uniqueness, extinction, and stability of the solution of the general SEIR model after incorporating interference from the transmission function and white noise. Finally, the conclusions of our theorems are verified through numerical simulations, which have certain reference significance for the future research of infectious disease models [28–30].

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References

1. Carcione, J.M.; Santos, J.E.; Bagaini, C.; Ba, J. A simulation of a COVID-19 epidemic based on a deterministic SEIR model. *Front. Public Health* **2020**, *8*, 230. [CrossRef] [PubMed]
2. Yang, Z.; Zeng, Z.; Wang, K.; Wong, S.-S.; Liang, W.; Zanin, M.; Liu, P.; Cao, X.; Gao, Z.; Mai, Z.; et al. Modified SEIR and AI prediction of the epidemics trend of COVID-19 in China under public health interventions. *J. Thorac. Dis.* **2020**, *12*, 165. [CrossRef] [PubMed]
3. Zhang, J.; Ma, Z. Global dynamics of an SEIR epidemic model with saturating contact rate. *Math. Biosci.* **2003**, *185*, 15–32. [CrossRef] [PubMed]
4. Piovella, N. Analytical solution of SEIR model describing the free spread of the COVID-19 pandemi. *Chaos Solitons Fractals* **2020**, *140*, 110243. [CrossRef] [PubMed]
5. Kishore, R.; Sahoo, B.; Swain, D.; Sahu, K.K. Analysis of COVID19 Outbreak in India using SEIR model. *arXiv* **2020**, arXiv:2010.13610.
6. Rohith, G.; Devika, K.B. Dynamics and control of COVID-19 pandemic with nonlinear incidence rates. *Nonlinear Dyn.* **2020**, *101*, 2013–2026. [CrossRef] [PubMed]
7. Kumar, G.; Ramesh, K.; Nisar, K.S. Dynamics of an SEIR epidemic model with saturated incidence rate including stochastic influence. *Comput. Methods Differ. Equ.* **2024**, *12*, 250–360.
8. Beretta, E.; Breda, D. An SEIR epidemic model with constant latency time and infectious period. *Math. Biosci. Eng.* **2011**, *8*, 931.
9. Rifhat, R.; Teng, Z.; Wang, C. Extinction and persistence of a stochastic SIRV epidemic model with nonlinear incidence rate. *Adv. Differ. Equ.* **2021**, *2021*, 200. [CrossRef]
10. Fan, X.; Wang, L.; Teng, Z. Global dynamics for a class of discrete SEIRS epidemic models with general nonlinear incidence. *Adv. Differ. Equ.* **2016**, *2016*, 123. [CrossRef]
11. Forien, R.; Pang, G.; Pardouxé, É. Epidemic models with varying infectivity. *Siam J. Appl. Math.* **2021**, *81*, 1893–1930. [CrossRef]

12. Shikongo, A.; Nuugulu, S.M.; Elago, D.; Salom, A.T.; Owolabi, K.M. Fractional Derivative Operator on Quarantine and Isolation Principle for COVID-19. *Adv. Numer. Methods Differ. Equ.* **2021**, *1*, 205–226.
13. Sun, C.; Lin, Y.; Tang, S. Global stability for an special SEIR epidemic model with nonlinear incidence rates. *Chaos Solitons Fractals* **2007**, *33*, 290–297. [CrossRef]
14. Naim, M.; Lahmidi, F.; Namir, A.; Kouidere, A. Dynamics of an fractional SEIR epidemic model with infectivity in latent period and general nonlinear incidence rate. *Chaos Solitons Fractals* **2021**, *152*, 111456. [CrossRef]
15. Jiao, J.; Liu, Z.; Cai, S. Dynamics of an SEIR model with infectivity in incubation period and homestead-isolation on the susceptible. *Appl. Math. Lett.* **2020**, *107*, 106442. [CrossRef]
16. Gourieroux, C.; Lu, Y. SIR model with stochastic transmission. *arXiv* **2020**, arXiv:2011.07816.
17. Yang, Q.; Jiang, D.; Shi, N.; Ji, C. The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence. *J. Math. Anal. Appl.* **2012**, *388*, 248–271. [CrossRef]
18. Wei, F.; Xue, R. Stability and extinction of SEIR epidemic models with generalized nonlinear incidence. *Math. Comput. Simul.* **2020**, *170*, 1–15. [CrossRef]
19. Han, X.; Li, F.; Meng, X. Dynamics analysis of a nonlinear stochastic SEIR epidemic system with varying population size. *Entropy* **2018**, *20*, 376. [CrossRef] [PubMed]
20. Wang, F.; Wang, S.; Peng, Y. Asymptotic Behavior of Multigroup SEIR Model with Nonlinear Incidence Rates under Stochastic Perturbations. *Discret. Dyn. Nat. Soc.* **2020**, *2020*, 1–12. [CrossRef]
21. Li, R.; Guo, X. Dynamics of a Stochastic SEIR Epidemic Model with Vertical Transmission and Standard Incidence. *Mathematics* **2024**, *12*, 359. [CrossRef]
22. Li, S. SIR Epidemic Model with General Nonlinear Incidence Rate and Lévy Jumps. *Mathematics* **2024**, *12*, 215. [CrossRef]
23. Vanden, D.P.; Watmough, J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathmetical Biosci.* **2002**, *180*, 29–48. [CrossRef] [PubMed]
24. Mao, X.R. *The Stability of Dynamical Systems*; Regional Conference Series in Applied Mathematics; SIAM: Philadelphia, PA, USA, 1976.
25. Mao, X.R.; Marion, G.; Renshaw, E. Environmental Brownian noise suppresses explosion in population dynamics. *Stoch. Process. Their Appl.* **2002**, *97*, 95–110. [CrossRef]
26. Mao, X.R. *Stochastic Differential Equations and Their Applications*; Horwood: Chichester, UK, 1997.
27. Higham, D.J. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev.* **2001**, *43*, 525–546. [CrossRef]
28. Chongyang, L.; Jie, G.; Jeevan, K. Dynamics analysis and optimal control of delayed SEIR model in COVID-19 epidemic. *J. Inequalities Appl.* **2024**, *2024*, 66.
29. Nabti, A. Dynamical analysis of an age-structured SEIR model with relapse. *Z. Angew. Math. Phys.* **2024**, *75*, 84. [CrossRef]
30. Chladná, Z.; Kopfová, J.; Rachinskii, D.; Štepánek, P. Effect of quarantine strategies in a compartmental model with asymptomatic groups. *J. Dyn. Differ. Equ.* **2024**, *36* (Suppl. S1), 199–222. [CrossRef]