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New Results on the Solvability of Abstract Sequential Caputo Fractional Differential Equations with a Resolvent-Operator Approach and Applications

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Abstract: This paper aims to establish the existence and uniqueness of mild solutions to abstract sequential fractional differential equations. The approach employed involves the utilization of resolvent operators and the fixed-point theorem. Additionally, we investigate a specific example concerning a partial differential equation incorporating the Caputo fractional derivative.

Keywords: Caputo fractional derivative; abstract sequential fractional differential equation; resolvent operators; nonlocal condition

MSC: 26A33; 34A08; 34B15

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1. Introduction

Fractional calculus is a more advanced version of traditional calculus and has a wider range of applications. It has been particularly useful in areas such as signal processing, chemistry, biology, control theory, physics, economic systems and mechanics [1–7].

In the field of biology, the authors of [8] utilized the Caputo fractional derivative as a mathematical technique to develop a model for the transmission of a coronavirus (specifically, MERS-CoV) between humans and camels. Camels are suspected to be the primary source of the infection. The paper investigates how the transmission of MERS-CoV disease changes over time by employing a nonlinear fractional order based on the Caputo operator. In the realm of physics, the nonlinear space–time fractional partial differential symmetric regularized long-wave equation is a useful tool for summarizing various physical phenomena. For example, it can describe ion-acoustic waves in plasma, as well as solitary waves and shallow-water waves. In [9], the authors used this novel approach to obtain the traveling wave solutions of two equations: the space–time fractional Cahn–Hilliard equation and the space–time fractional symmetric regularized long-wave equation.

Over the past two decades, several researchers have developed the theory of abstract impulsive and abstract fractional differential equations with nonlocal conditions; see, for instance, references [10–23] and the studies cited therein.

In [14], Hernandez et al. investigated the existence and uniqueness of a specific problem, defined as follows:

$$\begin{cases} \mathcal{D}^{\alpha}\chi(\kappa) = \mathcal{A}\chi(\kappa) + \varsigma(\kappa, \mathcal{B}\chi(\kappa), \chi(\kappa)), \ \kappa \in [0, a], \\ \chi(0) = \chi_0 + g(\chi), \end{cases} \tag{1}$$

where $\mathcal{D}^{\alpha}$ represents the Caputo fractional derivative, $\mathcal{A}$ is a closed linear operator with a domain contained in a Banach space X, and ς and g are continuous functions. The researchers employed various techniques, including the use of the resolvent operator and other properties of fractional differential equations, to study this problem.

In [24–34], the uniqueness and existence of solutions for a sequential fractional differential equation of the general form

$$\left(\mathcal{D}^{\alpha}+\lambda\mathcal{D}^{\beta}\right)\chi(\kappa)=\varsigma(\kappa),\ \lambda\in\mathbb{R}, \tag{2}$$

were examined, where $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$ are two fractional derivatives, and ς is a continuous function.

In [29], Aqlan et al. investigated the following sequential fractional equation:

$$\left(\mathcal{D}^{\alpha}+\lambda\mathcal{D}^{\alpha-1}\right)\chi(\kappa)=\varsigma(\kappa,\chi(\kappa)),\ \kappa\in[0,T],1<\alpha\leq 2,\ \lambda\in\mathbb{R}, \tag{3}$$

with boundary conditions of the form

$$\alpha_1\chi(0)+\rho_1\chi(T)=\beta_1,\alpha_2\chi'(0)+\rho_2\chi'(T)=\beta_2, \tag{4}$$

and with the nonlocal integral boundary conditions

$$\alpha_1\chi(0)+\rho_1\chi(T)=\lambda_1\int_0^a\chi(s)ds+\lambda_2,\alpha_2\chi'(0)+\rho_2\chi'(T)=\mu_1\int_{\xi}^{T}\chi(s)ds+\mu_2,$$

where $\mathcal{D}^{\alpha}$ is the Liouville–Caputo fractional derivative, and ς is a continuous function.

Salem and Almaghamsi [33] studied the existence of the solution of the following sequential fractional differential equation:

$$\mathcal{D}^{\alpha}(\mathcal{D}+\lambda)\chi(\kappa)=\varsigma(\kappa,\chi(\kappa),\chi'(\kappa),\mathcal{D}^{\alpha-1}\chi(\kappa))+e(\kappa),\kappa\in[0,1],\ \ \lambda\in\mathbb{R}, \tag{5}$$

with the boundary conditions

$$\chi(0)=0,\ \chi'(0)=0,\ \chi(1)=\beta\chi(\eta),\ 0<\eta<1, \tag{6}$$

where $1<\alpha\leq 2$, $\mathcal{D}^{\alpha}$ represents the Caputo derivative, and $\mathcal{D}$ denotes the first-order derivative.

In this paper, we extend Equations (2), (3) and (5) by considering the case where λ represents a closed linear operator $\mathcal{A}$. We investigate the following problem with an abstract sequential fractional differential equation of the form

$$\begin{cases}\mathcal{D}^{\beta}(\mathcal{D}^{\alpha}-\mathcal{A})\chi(\kappa)=\mathcal{H}_{\omega}(\kappa,\mathcal{I}^{\sigma}(\chi(\kappa)),\chi(\kappa)),\ \kappa\in[0,T],\\ \chi(0)=g(\chi),\\ \chi(T)=\mathcal{I}_T^{\alpha}(\mathcal{A}\chi),\end{cases} \tag{7}$$

where $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$ are two Caputo fractional derivatives, and $\mathcal{I}^{\sigma}$ is the Riemann–Liouville fractional integral, with $0<\alpha<1, 0<\beta,\sigma<1$. $\mathcal{A}$ is a closed linear unbounded operator, with the domain $\mathcal{D}(\mathcal{A})$ contained in a Banach space X, and $\mathcal{H}_{\omega}$ depends on a parameter $\omega\geq 0$, with $\mathcal{H}_{\omega}:[0,T]\times X^2\to X$, $g:C(J,X)\to X$, $J\subset\mathbb{R}$ being continuous functions.

Equation (7) can be interpreted as an abstract form of the following partial fractional differential equation:

$$\begin{cases}\dfrac{\partial^{\beta}}{\partial\kappa^{\beta}}\left(\dfrac{\partial^{\alpha}}{\partial\kappa^{\alpha}}-\dfrac{\partial^2}{\partial\tau^2}\right)\chi(\kappa,\tau)=\mathcal{H}_{\omega}(\kappa,\mathcal{I}^{\sigma}(\chi(\kappa,\tau)),\chi(\kappa,\tau)),\ \kappa\in[0,1],\\ \chi(\kappa,0)=\chi(\kappa,\pi)=0,\ \kappa\in[0,1],\\ \chi(0,\tau)=g(\chi(0,\tau)),\ \chi(1,\tau)=\mathcal{I}_1^{\frac{1}{2}}\left(\dfrac{\partial^2\chi(\kappa,\tau)}{\partial\tau^2}\right),\tau\in[0,\pi].\end{cases} \tag{8}$$

with $\mathcal{A}\chi=\chi_{\tau\tau}$ and $\mathcal{D}(\mathcal{A})=\{\chi\in\tau,\chi_{\tau\tau}\in\tau,\chi(\kappa,0)=\chi(\kappa,\pi)=0\}$, The main objective of this paper is to examine the existence and uniqueness of mild solutions of (1). In our study, we use the Caputo fractional derivative and Riemann–Liouville fractional integral operators, with a specific emphasis on the significance of resolvent operators. To demonstrate the uniqueness, we use the Banach contraction principle, and for the existence of solutions, we apply the Krasnoseskii fixed-point theorem.

In our study, the domain of the operator $\mathcal{A}$, represented by $\mathcal{D}(\mathcal{A})$, is equipped with the graph norm, where $\|\chi\|_{\mathcal{D}(\mathcal{A})} = \|\chi\| + \|\mathcal{A}\chi\|$. The norm of the space $C([0,T],X)$ is defined by $\|\chi\|_{C([0,T],X)} = \max\limits_{\kappa\in[0,T]} \|\chi(\kappa)\|$.

2. Preliminaries

In this section, we will present definitions and preliminary concepts that will serve as building blocks for the next sections. These essential definitions and preliminary explanations will be referred to frequently in the upcoming sections.

Definition 1 ([6])**.** *Let $\alpha \in \mathbb{R}$ such that $n-1 < \alpha < n, n \in \mathbb{N}^*$. The Caputo fractional derivative of order α for a function $\varsigma \in C^n(0,\infty)$ is defined by*

$$\begin{aligned}\mathcal{D}^\alpha[\varsigma(\kappa)] &= \frac{1}{\Gamma(n-\alpha)} \int_{\mathcal{A}}^{\kappa} (\kappa - s)^{n-\alpha-1} \varsigma^{(n)}(s) ds \\ &= \mathcal{I}^{n-\alpha} \varsigma^{(n)}(s),\end{aligned}$$

for $\kappa > 0$.

Definition 2 ([6])**.** *The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function ς is defined by*

$$\mathcal{I}_\kappa^\alpha[\varsigma(\kappa)] = \frac{1}{\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \varsigma(s) ds,$$

for $\kappa > 0$, where the Gamma function $\Gamma(\alpha) = \int_0^\infty e^{-\tau} \tau^{\alpha-1} d\tau$.

Now, we present the following lemmas (the interested reader is referred to [35] for more details).

Lemma 1 ([35])**.** *Let $\alpha, \beta > 0, \varsigma \in L^1([0,T])$. Then,*

$$\mathcal{I}^\alpha \mathcal{I}^\beta \varsigma(\kappa) = \mathcal{I}^{\alpha+\beta} \varsigma(\kappa), \mathcal{D}^\beta \mathcal{I}^\beta \varsigma(\kappa) = \varsigma(\kappa), \kappa \in [0,T]. \tag{9}$$

Lemma 2 ([35])**.** *Let $\beta > \alpha > 0, \varsigma \in L^1([0,T])$. Then,*

$$\mathcal{D}^\alpha \mathcal{I}^\beta \varsigma(\kappa) = \mathcal{I}^{\beta-\alpha} \varsigma(\kappa), \kappa \in [0,T]. \tag{10}$$

To establish our results, we assume that the integral equation

$$\chi(\kappa) = \frac{1}{\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \mathcal{A}\chi(s) ds, \kappa \geq 0, \tag{11}$$

has an associated resolvent operator $(\mathcal{R}(\kappa))_{\kappa\geq 0}$ on X.

In the following two definitions, we will present some results that can be found in [36].

Definition 3 ([36])**.** *A one-parameter bounded linear operator $\{\mathcal{R}(\kappa)\}_{\kappa\geq 0}$ on X is called a resolvent operator for (11) if it fulfills the following conditions:*

1. *$\mathcal{R}(\cdot)\chi \in C([0,\infty), X)$ and $\mathcal{R}(0)\chi = \chi$ for all $\chi \in X$.*
2. *$\mathcal{R}(\kappa)\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\mathcal{R}(\kappa)\chi = \mathcal{R}(\kappa)\mathcal{A}\chi$ for all $\chi \in \mathcal{D}(\mathcal{A})$ and $\kappa \geq 0$.*

3. *For all* $\chi \in \mathcal{D}(\mathcal{A})$ *and* $\kappa \geq 0$,

$$\mathcal{R}(\kappa)\chi = \chi + \frac{1}{\Gamma(\alpha)} \int_0^{\kappa} (\kappa - s)^{\alpha-1} \mathcal{A}\mathcal{R}(s)\chi ds.$$

Definition 4 ([36])**.** *A resolvent operator* $(\mathcal{R}(\kappa))_{\kappa \geq 0}$ *for (11) is called differentiable if the following conditions are satisfied:*

1. $\mathcal{R}(\kappa)\chi \in W^{1,1}_{loc}(\mathbb{R}^+, X)$ *for all* $\chi \in \mathcal{D}(\mathcal{A})$.
2. *There exists* $\phi_{\mathcal{A}} \in L^1_{loc}(\mathbb{R}^+)$ *such that* $\|\mathcal{R}'(\kappa)\chi\| \leq \phi_{\mathcal{A}} \|\chi\|_{\mathcal{D}(\mathcal{A})}$ *for all* $\chi \in \mathcal{D}(\mathcal{A})$.

Definition 5 ([36])**.** *Let us consider the integral equation*

$$\chi(\kappa) = \frac{\mathcal{A}}{\Gamma(\alpha)} \int_0^{\kappa} (\kappa - s)^{\alpha-1} \chi(s) ds + f(\kappa), \ \kappa \in [0, T]. \tag{12}$$

Then, the function $\psi \in C([0, T], X)$ *is called a mild solutionof the introduced integral Equation (12) if*

$$\int_0^{\kappa} (\kappa - s)^{\alpha-1} \psi(s) ds \in \mathcal{D}(\mathcal{A}) \quad \textit{for all} \quad \kappa \in [0, T], f \in L^1([0, T], X).$$

and Equation (12) is satisfied.

Lemma 3 ([36])**.** *Under the conditions stated in Definition 3, the following properties hold true:*

1. *If* χ *is a mild solution of (12) on* $[0, T]$, *then the function* $\kappa \rightarrow \int_0^{\kappa} \mathcal{R}(\kappa - s) f(s) ds$ *is continuously differentiable on* $[0, T]$ *and*

$$\chi(\kappa) = \frac{d}{d\kappa} \int_0^{\kappa} \mathcal{R}(\kappa - s) f(s) ds, \ \kappa \in [0, T].$$

2. *If* $(\mathcal{R}(\kappa))_{\kappa \geq 0}$ *is differentiable and* $f \in C([0, T], \mathcal{D}(\mathcal{A}))$, *then the function* $\chi : [0, T] \rightarrow X$ *defined by*

$$\chi(\kappa) = \int_0^{\kappa} \mathcal{R}'(\kappa - s) f(s) ds + f(\kappa), \ \kappa \in [0, T],$$

is a mild solution of (12) on $[0, T]$.

For the proof of this lemma, refer to [36]. The following example shows how operators and their properties are used.

Example 1. *Let us consider the following abstract fractional problem:*

$$\begin{cases} \mathcal{D}^{\delta}(\chi(\kappa)) = \mathcal{B}\chi(\kappa) + \varsigma(\kappa), \ \kappa \in [0, T], \\ \chi(0) = \chi_0, \end{cases} \tag{13}$$

where $\mathcal{D}^{\delta}$ *is the Caputo fractional derivative,* $0 < \delta < 1$, $\mathcal{B}$ *is a closed linear operator,* ς *is a continuous function, and* $\chi_0 \in X$.

Problem (13) is equivalent to

$$\chi(\kappa) = \frac{1}{\Gamma(\delta)} \int_0^{\kappa} (\kappa - s)^{(\delta-1)} \mathcal{B}\chi(s) ds + \frac{1}{\Gamma(\delta)} \int_0^{\kappa} (\kappa - s)^{(\delta-1)} \varsigma(s) ds + \chi_0.$$

The equation mentioned above can be expressed as an integral equation in thefollowing form:

$$\chi(\kappa) = \frac{1}{\Gamma(\delta)} \int_0^{\kappa} (\kappa - s)^{(\delta-1)} \mathcal{B}\chi(s) ds + f(\kappa), \ \kappa \geq 0, \tag{14}$$

where $f(\kappa) = \chi_0 + \frac{1}{\Gamma(\delta)} \int_0^{\kappa} (\kappa - s)^{(\delta-1)} \varsigma(s) ds$.

Assuming the existence of a differentiable resolvent operator $\mathcal{S}(\kappa), \kappa \geq 0$, for Problem (13), then by using Point (2) of Lemma 3, we can write

$$\chi(\kappa) = \chi_0 + \frac{1}{\Gamma(\delta)} \int_0^{\kappa} (\kappa - s)^{(\delta-1)} \varsigma(s) ds$$
$$+ \int_0^{\kappa} \mathcal{S}'(\kappa - s) \left(\chi_0 + \frac{1}{\Gamma(\delta)} \int_0^{s} (s - \tau)^{(\delta-1)} \varsigma(\tau) d\tau \right) d\kappa, \kappa \in [0, T]. \tag{15}$$

We present two fixed-point theorems that allow us to establish the uniqueness and existence results, as mentioned in references [37,38].

Theorem 1 (Banach's fixed-point theorem). *Let Ω be a nonempty closed subset of a Banach space X; then, any contraction mapping Ψ of Ω onto itself has a unique fixed point.*

Theorem 2 (Krasnoselskii fixed-point theorem). *Let Ω be a closed convex and nonempty subset of a Banach space X. Let Ψ_1 and Ψ_2 be two operators such that*

1. $\Psi_1 x + \Psi_2 y \in \Omega$, *with* $x, y \in \Omega$.
2. Ψ_1 *is contraction.*
3. Ψ_2 *is compact and continuous.*

Then, there exists $z \in \Omega$ such that$z = \Psi_1 z + \Psi_2 z$.

This paper is organized as follows.

In Section 3, we examine the existence of mild solutions and establish the theorems regarding the existence and uniqueness of the mild solution to Problem (1). Section 4 presents the results concerning the existence in the specific case of $\mathcal{A} \equiv \lambda$, $\lambda \in \mathbb{R}$. In Section 5, we investigate an example of partial differential equations with the Caputo fractional derivative.

3. Main Results

In this section, we investigate the existence of mild solutions to Problem (1). We make the following assumptions throughout this study:

Hypothesis 1 (H1). *$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ is a closed linear operator.*

Hypothesis 2 (H2). *The resolvent operator $\mathcal{R}(t)$, $t \geq 0$, is differentiable, and there exists a function $\phi_{\mathcal{A}}$ in $L^1_{loc}([0, \infty); \mathbb{R}^+)$ such that*

$$\|\mathcal{R}'(t)x\| \leq \phi_{\mathcal{A}}(t) \|x\|_{\mathcal{D}(\mathcal{A})}, \text{ for all } t > 0.$$

Hypothesis 3 (H3). *For $\omega \geq 0$, the function $\mathcal{H}_\omega : [0, T] \times X^2 \to X$ is completely continuous, and there exists a constant $L_\omega > 0$ such that*

For all $(\kappa, \tau_i, s_i) \in [0, T] \times X^2$, $i = 1, 2$, we have

$$\|\mathcal{H}_\omega(\kappa, \tau_1, s_1) - \mathcal{H}_\omega(\kappa, \tau_2, s_2)\| \leq L_\omega(\|\tau_1 - \tau_2\| + \|s_1 - s_2\|) .$$

Let us consider

$$M = \max\{\mathcal{H}_\omega(\kappa, 0, 0),\ \kappa \in [0, T]\}.$$

Hypothesis 4 (H4). *There exists $\omega^* \geq 0, \forall \omega \geq \omega^*$, and we have*

$$\|g(\tau_1) - g(\tau_2)\| \leq \rho(\omega) \|\tau_1 - \tau_2\|, \forall \tau_i \in X,\ i = 1, 2,$$

where $\rho : [w^*, +\infty] \to \mathbb{R}^+$ is such that $\lim\limits_{\omega \to +\infty} \rho(\omega) = 0$.

Hypothesis 5 (H5). *There exists $\omega^* \geq 0$, for all $\omega \geq \omega^*$, such that*

$$\Upsilon = 2(1 + \|\phi_{\mathcal{A}}\|_{L^1})\left(\frac{2L_\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right) + \rho(\omega)\right) < 1.$$

Lemma 4. *Let us consider the problem*

$$\begin{cases} \mathcal{D}^\beta(\mathcal{D}^\alpha - \mathcal{A})\chi(\kappa) = h(\kappa),\ \kappa \in [0,T], \\ \chi(0) = g(\chi),\ \chi(T) = \mathcal{I}_T^\alpha(\mathcal{A}\chi) \end{cases} \tag{16}$$

where $h \in C([0,T], X), 0 < \alpha, \beta < 1$.

Then, Problem (16) is equivalent to

$$\begin{aligned} \chi(\kappa) = &\frac{1}{\Gamma(\alpha)}\int_0^\kappa (\kappa-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}h(s)ds\right)d\tau \\ &- \frac{\kappa^\alpha}{T^\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}h(s)ds\right)d\tau\right) \\ &+ g(\chi)\left[1 - \frac{\kappa^\alpha}{T^\alpha}\right] + \frac{1}{\Gamma(\alpha)}\int_0^\kappa (\kappa-\tau)^{\alpha-1}\mathcal{A}\chi(\tau)d\tau. \end{aligned}$$

Proof. We have

$$\mathcal{D}^\beta(\mathcal{D}^\alpha - \mathcal{A})\chi(\kappa) = h(\kappa). \tag{17}$$

By applying the Riemann–Liouville fractional integral of order β toEquation (17), we obtain

$$\mathcal{D}^\alpha\chi(\kappa) - \mathcal{A}\chi(\kappa) = \frac{1}{\Gamma(\beta)}\int_0^\kappa (\kappa-s)^{\beta-1}h(s)ds + c_0. \tag{18}$$

By once again applying the Riemann–Liouville fractional integral of orderα to Equation (18), we obtain the following result:

$$\begin{aligned} \chi(\kappa) = &\frac{1}{\Gamma(\alpha)}\int_0^\kappa (\kappa-\tau)^{\alpha-1}\mathcal{A}\chi(\tau)d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^\kappa (\kappa-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}h(s)ds\right)d\tau \\ &+ \frac{c_0}{\Gamma(\alpha+1)}\kappa^\alpha + c_1, \end{aligned}$$

where c_0, c_1 are constants.

Using the first boundary condition $\chi(0) = g(\chi)$, we obtain $c_1 = g(\chi)$. The second boundary condition, $\chi(T) = \mathcal{I}_T^\alpha(\mathcal{A}\chi)$, implies

$$\begin{aligned} \chi(T) = &\frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\mathcal{A}\chi(\tau)d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}h(s)ds\right)d\tau \\ &+ \frac{c_0}{\Gamma(\alpha+1)}T^\alpha + g(\chi) \\ = &\frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\mathcal{A}\chi(\tau)d\tau, \end{aligned}$$

which means

$$c_0 = \frac{\Gamma(\alpha+1)}{T^\alpha}\left(-g(\chi) - \frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}h(s)ds\right)d\tau\right).$$

Consequently, Problem (16) is equivalent to the following:

$$\begin{aligned}\chi(\kappa) = \frac{1}{\Gamma(\alpha)}\int_0^{\kappa}(\kappa-\tau)^{\alpha-1}\mathcal{A}\chi(\tau)d\tau \\ + \frac{1}{\Gamma(\alpha)}\int_0^{\kappa}(\kappa-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-s)^{\beta-1}h(s)ds\right)d\tau \\ - \frac{\kappa^{\alpha}}{T^{\alpha}\times\Gamma(\alpha)}\int_0^{T}(T-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-s)^{\beta-1}h(s)ds\right)d\tau \\ + g(\chi)\left[1-\frac{\kappa^{\alpha}}{T^{\alpha}}\right].\end{aligned} \quad (19)$$

□

Let us denote the solution of Problem (16) as follows:

$$\chi(\kappa) = \mathcal{I}_{\kappa}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}h(s)\right) - \frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}h(s)\right) + g(\chi)\left(1-\frac{\kappa^{\alpha}}{T^{\alpha}}\right) + \mathcal{I}_{\kappa}^{\alpha}(\mathcal{A}\chi(\tau)).$$

Remark 1. *Equation (19) can be alternatively represented as an integral equation in the following scientific form:*

$$\chi(\kappa) = \frac{1}{\Gamma(\alpha)}\int_0^{\kappa}(\kappa-\tau)^{\alpha-1}\mathcal{A}\chi(\tau)d\tau + f(\kappa),\ \kappa \geq 0, \quad (20)$$

where

$$f(\kappa) = \mathcal{I}_{\kappa}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}h(s)\right) - \frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}h(s)\right) + g(\chi)\left(1-\frac{\kappa^{\alpha}}{T^{\alpha}}\right).$$

Using Lemma 4, we can establish the equivalence of Problem (1) to the following integral equation:

$$\begin{aligned}\chi(\kappa) = \mathcal{I}_{\kappa}^{\alpha}(\mathcal{A}\chi(\tau)) + \mathcal{I}_{\kappa}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\right) \\ - \frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\right) + g(\chi)\left(1-\frac{\kappa^{\alpha}}{T^{\alpha}}\right),\end{aligned} \quad (21)$$

where $\kappa \in [0,T]$.

In the subsequent definition, we present a conceptually similar definition for the mild solution of Problem (1).

Definition 6. *A function $\chi \in C([0,T],X)$ is considered to be a mild solution of Problem (1) in the interval $[0,T]$ if $\mathcal{I}_{\kappa}^{\alpha}(\chi(\tau)) = \frac{1}{\Gamma(\alpha)}\int_0^{\kappa}(\kappa-\tau)^{\alpha-1}\chi(\tau)d\tau \in \mathcal{D}(\mathcal{A})$ for all $\kappa \in [0,T]$ and*

$$\begin{aligned}\chi(\kappa) = \mathcal{A}(\mathcal{I}_{\kappa}^{\alpha}(\chi(\tau))) + \mathcal{I}_{\kappa}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\right) \\ - \frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha}\left(\mathcal{I}_{\tau}^{\beta}\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\right) + g(\chi)\left(1-\frac{\kappa^{\alpha}}{T^{\alpha}}\right), \kappa \in [0,T].\end{aligned}$$

Now, we present the main theorems, along with their corresponding proofs, regarding the existence and uniqueness of the mild solution to Problem (1).

3.1. The Uniqueness of the Mild Solution

Theorem 3. *Under assumptions **(H1)–(H5)**, there exists a unique mild solution of Problem (1) in the interval $[0,T]$.*

Proof. Suppose there exists a differentiable resolvent operator $\mathcal{R}(\kappa)$ for $\kappa \geq 0$, and the functions $\mathcal{H}_\omega$ and g are continuous in X. Additionally, referring to Remark 1 and Property 2 of Lemma 3, we define the map $\Lambda : C([0,T],X) \to C([0,T],X)$, for $\omega \geq 0$, by

$$\begin{aligned}\Lambda\chi(\kappa) = \int_0^\kappa \mathcal{R}'(\kappa - s)\Big(&\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big) \\ &-\frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) + g(\chi)\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)\Big)ds \\ &+ \mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) - \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) \\ &+ g(\chi)\Big(1 - \frac{\kappa^\alpha}{T^\alpha}\Big).\end{aligned} \tag{22}$$

The goal now is to prove that Λ is a contraction.

Let $\chi \in C([0,T],X)$; then, from the assumption on $\mathcal{H}_\omega$, for $\omega \geq 0$, we have

$$\begin{aligned}&\Big\|\int_0^\kappa \mathcal{R}'(\kappa - s)\Big(\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big) \\ &\qquad -\frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) + g(\chi)\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)\Big)ds\Big\| \\ &\leq \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\Big(\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta(1)\Big) + \frac{s^\alpha}{T^\alpha}\mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta(1)\Big)\Big)\sup_{t\in[0,T]}\|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\|ds \\ &\qquad + \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\|g(\chi)\|\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)ds \\ &\leq \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\Big(\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta(1)\Big) + \frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta(1)\Big)\Big)ds \times \sup_{t\in[0,T]}\|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\| \\ &\qquad + \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\|g(\chi)\|\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)ds \\ &\leq \|\phi_\mathcal{A}\|_{L^1}\Big(2\frac{\kappa^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\sup_{t\in[0,T]}\|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\| + \|g(\chi)\|\Big).\end{aligned}$$

Then, we have

$$\begin{aligned}s \to \mathcal{R}'(\kappa - s)\Big(&\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big) \\ &-\frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) + g(\chi)\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)\Big)\end{aligned}$$

which is integrable on $[0,T]$ for all $\kappa \in [0,T]$.

This leads to the conclusion that $\Lambda\chi \in C([0,T],X)$, and as a result, Λ is well defined. Moreover, for $\chi, \psi \in C([0,T],X)$, we have

$$
\begin{aligned}
&\|\Lambda\chi-\Lambda\psi\| \\
&\quad\leq \mathcal{I}_{\kappa}^{\alpha}\Big[\mathcal{I}_{\tau}^{\beta}(\|\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))-\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\psi(s),\psi(s)))\|\Big] \\
&\qquad+\frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha}\Big(\mathcal{I}_{\tau}^{\beta}(\|\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))-\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\psi(s),\psi(s))\|)\Big) \\
&\qquad+\|g(\chi)-g(\psi)\|\left(1-\frac{\kappa^{\alpha}}{T^{\alpha}}\right) \\
&\qquad+\int_0^{\kappa}\|\mathcal{R}'(\kappa-s)\|\Big(\mathcal{I}_{s}^{\alpha}\Big(\mathcal{I}_{\tau}^{\beta}(\|\mathcal{H}_{\omega}(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))-\mathcal{H}_{\omega}(t,\mathcal{I}^{\sigma}\psi(t),\psi(t))\|\Big) \\
&\qquad+\frac{s^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha}\Big(\mathcal{I}_{\tau}^{\beta}(\|\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))-\mathcal{H}_{\omega}(s,\mathcal{I}^{\sigma}\psi(s),\psi(s))\|\Big)\Big)d\kappa \\
&\qquad+\int_0^{\kappa}\|\mathcal{R}'(\kappa-s)\|\|g(\chi)-g(\psi)\|\left(1-\frac{s^{\alpha}}{T^{\alpha}})\right)d\kappa,
\end{aligned}
$$

and by using hypotheses **(H2)**, **(H3)** and **(H4)**, we obtain

$$
\begin{aligned}
&\|\Lambda\chi-\Lambda\psi\| \\
&\quad\leq \left(\mathcal{I}_{\kappa}^{\alpha+\beta}(1)+\frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha+\beta}(1)\right)L_{\omega}(\|\mathcal{I}_{\kappa}^{\sigma}\chi(t)-\mathcal{I}_{\kappa}^{\sigma}\psi(t)\|+\|\chi-\psi\|) \\
&\qquad+\rho(\omega)\|\chi-\psi\|+\left(\int_0^{\kappa}\phi_{\mathcal{A}}\left(\mathcal{I}_{\kappa}^{\alpha+\beta}(1)+\frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}_{T}^{\alpha+\beta}(1)\right)ds\right) \\
&\qquad\quad\times L_{\omega}(\|\mathcal{I}_{\kappa}^{\sigma}\chi(t)-\mathcal{I}_{\kappa}^{\sigma}\psi(t)\|+\|\chi-\psi\|) \\
&\qquad+\|\phi_{\mathcal{A}}\|_{L^1}\rho(\omega)\|\chi-\psi\|.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|\Lambda\chi-\Lambda\psi\| &\leq 2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}L_{\omega}(\|\mathcal{I}_{\kappa}^{\sigma}\chi(t)-\mathcal{I}_{\kappa}^{\sigma}\psi(t)\|+\|\chi-\psi\|)+\rho(\omega)\|\chi-\psi\| \\
&+2\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}L_{\omega}(\|\mathcal{I}_{\kappa}^{\sigma}\chi(t)-\mathcal{I}_{\kappa}^{\sigma}\psi(t)\|+\|\chi-\psi\|)\|\phi_{\mathcal{A}}\|_{L^1}+\rho(\omega)\|\chi-\psi\|\|\phi_{\mathcal{A}}\|_{L^1}.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
\|\mathcal{I}_{\kappa}^{\sigma}\chi(t)-\mathcal{I}_{\kappa}^{\sigma}\psi(t)\| &= \left\|\frac{1}{\Gamma(\sigma)}\int_0^{\kappa}(\kappa-s)^{\sigma-1}\chi(s)ds-\frac{1}{\Gamma(\sigma)}\int_0^{\kappa}(\kappa-s)^{\sigma-1}\psi(s)ds\right\| \\
&\leq \frac{1}{\Gamma(\sigma)}\int_0^{\kappa}(\kappa-s)^{\sigma-1}\|\chi(s)-\psi(s)\|ds \\
&\leq \frac{T^{\sigma}}{\Gamma(\sigma+1)}\|\chi(s)-\psi(s)\|,\ 0<\sigma<1,
\end{aligned}
$$

we finally obtain

$$
\begin{aligned}
\|\Lambda\chi-\Lambda\psi\| &\leq \left(2\frac{L_{\omega}T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\frac{T^{\sigma}}{\Gamma(\sigma+1)}+1\right)+\rho(\omega)\right)\|\chi-\psi\| \\
&\quad+\left(2\frac{L_{\omega}T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\frac{T^{\sigma}}{\Gamma(\sigma+1)}+1\right)+\rho(\omega)\right)\|\phi_{\mathcal{A}}\|_{L^1}\|\chi-\psi\| \\
&\leq (1+\|\phi_{\mathcal{A}}\|_{L^1})\left(2\frac{L_{\omega}T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\frac{T^{\sigma}}{\Gamma(\sigma+1)}+1\right)+\rho(\omega)\right)\|\chi-\psi\|.
\end{aligned}
$$

Due to assumption **(H5)**, there exists $\omega^*\geq 0$ such that for all $\omega\geq\omega^*$, the operator Λ is a contraction. By applying Banach's fixed-point theorem, we conclude that there exists a unique mild solution to Problem (1). Thus, the proof is complete. □

3.2. The Existence of the Mild Solution

Theorem 4 (Existence)**.** *Under assumptions* ***(H1)–(H5)****, there exists a mild solution to Problem (1) in the interval* $[0, T]$*.*

Proof. We convert the existence of a solution to Problem (1) into a fixed-point problem. We introduce a map denoted by $\Lambda : C([0, T], X) \to C([0, T], X)$, which is defined according to Equation (22), stated in the proof of the previous theorem.

$$\begin{aligned}\Lambda\chi(\kappa) = \int_0^\kappa \mathcal{R}'(\kappa - s)\Big(\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big)\\ -\frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big) + g(\chi)\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)\Big)ds\\ + \mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) - \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big)\\ + g(\chi)\Big(1 - \frac{\kappa^\alpha}{T^\alpha}\Big).\end{aligned}$$

We decompose Λ into two parts, denoted by Λ_1 and Λ_2, on the closed ball $\mathcal{B}_r(0, E)$, where $\mathcal{B}_r(0, E)$ represents the closed ball centered at 0 with the radius r in the space $E = C([0, T], X)$, where

$$\Lambda_1\chi(\kappa) = \int_0^\kappa \mathcal{R}'(\kappa - s)g(\chi)\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)ds + g(\chi)\Big(1 - \frac{\kappa^\alpha}{T^\alpha}\Big),$$

and

$$\begin{aligned}\Lambda_2\chi(\kappa) = \int_0^\kappa \mathcal{R}'(\kappa - s)\Big(\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big)\\ -\frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big)\Big)ds\\ + \mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) - \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big).\end{aligned}$$

Obviously, due to hypothesis **(H3)**, we have

$$\mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\Big) - \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\Big) \in C([0, T], X).$$

Let $\mathcal{B}_r(0, E) = \{z \in E = C([0, T], X) : \|z\| \le r\}$. For $\chi, \psi \in C([0, T], X)$. We choose

$$2(1 + \|\phi_{\mathcal{A}}\|_{L^1})\Big(2M\Big(\frac{T^\alpha}{\Gamma(\alpha + \beta + 1)}\Big) + \|g(0)\|\Big) < r. \tag{23}$$

Then, for $\chi, \psi \in \mathcal{B}_r(0, E)$ and $\omega \ge 0$, we have

$$\begin{aligned}\|\Lambda_1\chi(\kappa) + \Lambda_2\psi(\kappa)\|\\ \le \Big\|\int_0^\kappa \mathcal{R}'(\kappa - s)g(\chi)\Big(1 - \frac{s^\alpha}{T^\alpha}\Big)ds + \Big(1 - \frac{\kappa^\alpha}{T^\alpha}\Big)g(\chi)\Big\|\\ + \Big\|\int_0^\kappa \mathcal{R}'(\kappa - s)\Big(\mathcal{I}_s^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\psi(t), \psi(t))\Big)\\ -\frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(t, \mathcal{I}^\sigma\psi(t), \psi(t))\Big)\Big)ds\Big\|\\ + \Big\|\mathcal{I}_\kappa^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\psi(s), \psi(s))\Big) - \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\Big(\mathcal{I}_\tau^\beta\mathcal{H}_\omega(s, \mathcal{I}^\sigma\psi(s), \psi(s))\Big)\Big\|,\end{aligned}$$

and then, since $\omega \geq 0$, we obtain

$$\begin{aligned}
&\|\Lambda_1\chi(\kappa) + \Lambda_2\psi(\kappa)\| \\
&\quad \leq \left(\int_0^\kappa \|\mathcal{R}'(\kappa - s)\|(\|g(\chi) - g(0)\| + \|g(0)\|)ds + \|g(\chi) - g(0)\| + \|g(0)\|\right) \\
&\qquad + \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\Big\{\mathcal{I}_s^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\psi(t), \psi(t)) - \mathcal{H}_\omega(t, 0, 0)\|) \\
&\qquad + \frac{s^\alpha}{T^\alpha}\left(\mathcal{I}_s^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\psi(t), \psi(t)) - \mathcal{H}_\omega(t, 0, 0)\|)\right)\Big\}ds \\
&\qquad + \mathcal{I}_\kappa^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\psi(s), \psi(s)) - \mathcal{H}_\omega(s, 0, 0)\|) \\
&\qquad + \frac{\kappa^\alpha}{T^\alpha}\left(\mathcal{I}_\kappa^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\psi(s), \psi(s)) - \mathcal{H}_\omega(s, 0, 0)\|)\right) \\
&\qquad + \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\left[\mathcal{I}_s^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(t, 0, 0)\|) + \frac{s^\alpha}{T^\alpha}\left(\mathcal{I}_T^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(t, 0, 0)\|)\right)\right]ds \\
&\qquad + \mathcal{I}_\kappa^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, 0, 0)\|) + \frac{\kappa^\alpha}{T^\alpha}\left(\mathcal{I}_\kappa^\alpha \mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, 0, 0)\|)\right).
\end{aligned}$$

Using hypotheses **(H3)** and **(H4)**, we can deduce that

$$\begin{aligned}
&\|\Lambda_1\chi(\kappa) + \Lambda_2\psi(\kappa)\| \\
&\quad \leq \|\phi_\mathcal{A}\|_{L^1}(\rho(\omega)r + \|g(0)\|) + (\rho(\omega)r + \|g(0)\|) \\
&\qquad + \|\phi_\mathcal{A}\|_{L^1} L_\omega r\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right)\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{T^\alpha}\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\right) \\
&\qquad + L_\omega r\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right)\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{T^\alpha}\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&\qquad + \|\phi_\mathcal{A}\|_{L^1} M\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{T^\alpha}\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&\qquad + M\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^\alpha}{T^\alpha}\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|\Lambda_1\chi(\kappa) + \Lambda_2\psi(\kappa)\| \\
&\quad \leq (\|\phi_\mathcal{A}\|_{L^1} + 1)\rho(\omega)r + (\|\phi_\mathcal{A}\|_{L^1} + 1)\|g(0)\| \\
&\qquad + 2\|\phi_\mathcal{A}\|_{L^1} L_\omega r\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right)\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&\qquad + 2L_\omega r\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right)\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) \\
&\qquad + 2\|\phi_\mathcal{A}\|_{L^1} M\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) + 2M\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right).
\end{aligned}$$

So, for $\omega \geq 0$,

$$\begin{aligned}
\|\Lambda_1\chi(\kappa) + \Lambda_2\psi(\kappa)\| \leq (\|\phi_\mathcal{A}\|_{L^1} + 1)&\left(\rho(\omega) + 2\frac{L_\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right)\right)r \\
&+ (\|\phi_\mathcal{A}\|_{L^1} + 1)\left(2M\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right) + \|g(0)\|\right).
\end{aligned}$$

Then, due to hypothesis **(H5)** and the condition stated in (23), we have

$$\|\Lambda_1\chi(\kappa) + \Lambda_2\psi(\kappa)\| < r.$$

Consequently, for any χ and ψ in $\mathcal{B}_r(0,E)$, we have$\Lambda_1\chi + \Lambda_2\psi \in \mathcal{B}_r(0,E)$. From assumptions **(H2)** and **(H4)**, we can observe that for any $\chi \in C([0,T],X)$, the inequality

$$\left\|\int_0^\kappa \mathcal{R}'(\kappa - s)g(\chi)ds\right\| \leq \|\phi_\mathcal{A}\|_{L^1}(\rho(\omega)r + \|g(0)\|),$$

holds. Consequently, we can deduce that the function $s \to \mathcal{R}'(\kappa - s)g(\chi)$ is integrable over $[0,T]$ for all $\kappa \in [0,T]$, and therefore, $\Lambda_1\chi \in C([0,T],X)$. Moreover, for $\chi, \psi \in C([0,T],X)$ and $\kappa \in [0,T]$, we obtain, for $\omega \geq 0$,

$$\begin{aligned}
\|\Lambda_1\chi(\kappa) - \Lambda_1\psi(\kappa)\| \leq & \left|1 - \frac{\kappa^\alpha}{T^\alpha}\right| \|g(\chi) - g(\psi)\| \\
& + \int_0^\kappa \|\mathcal{R}'(\kappa - s)\| \left|1 - \frac{s^\alpha}{T^\alpha}\right| \|g(\chi) - g(\psi)\| ds \\
& \leq \rho(\omega)\|\chi - \psi\| + \|\phi_\mathcal{A}\|_{L^1}\rho(\omega)\|\chi - \psi\| \\
& \leq \rho(\omega)(1 + \|\phi_\mathcal{A}\|_{L^1})\|\chi - \psi\|.
\end{aligned}$$

By leveraging hypothesis **(H5)**, we can establish that Λ_1 is a contraction on $\mathcal{B}_r(0,E)$.

In this step, we will show that the operator Λ_2 is compact and continuous. Note that the function

$$s \to \int_0^\kappa \mathcal{R}'(\kappa - s)\left(\mathcal{I}_s^\alpha\left(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\right) - \frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\left(\mathcal{I}_\tau^\beta \mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\right)\right)ds,$$

is integrable based on assumptions **(H2)** and **(H3)**, as demonstrated earlier. First, we show that Λ_2 is uniformly bounded. Indeed, for $\kappa \in [0,T]$ and $\omega \geq 0$, we have

$$\begin{aligned}
\|\Lambda_2\chi(\kappa)\| \leq & \int_0^\kappa \mathcal{R}'(\kappa - s)\left(\mathcal{I}_s^\alpha\left(\mathcal{I}_\tau^\beta \|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\|\right)\right. \\
& \left. + \frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\left(\mathcal{I}_\tau^\beta \|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\|\right)\right)ds \\
& + \mathcal{I}_\kappa^{\alpha-\beta}\left(\mathcal{I}_\tau^\beta \|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\|\right) + \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\left(\mathcal{I}_\tau^\beta \|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\|\right) \\
\leq & \, 2(\|\phi_\mathcal{A}\|_{L^1} + 1)\left[\frac{rL_\omega T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left(\frac{T^\sigma}{\Gamma(\sigma+1)} + 1\right) + \frac{MT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right].
\end{aligned}$$

So, Λ_2 is uniformly bounded.

Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_r(0,E)$ such that $\chi_n \to \chi$ in $\mathcal{B}_r(0,E)$. Since the function $\mathcal{H}_\omega$ is continuous,

$$\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi_n(s), \chi_n(s)) \to \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s)) \text{ as } n \to \infty,$$

and

$$\begin{aligned}
& \|\Lambda_2\chi_n(\kappa) - \Lambda_2\chi(\kappa)\| \\
& \leq \int_0^\kappa \|\mathcal{R}'(\kappa - s)\|\left(\mathcal{I}_s^\alpha\left(\mathcal{I}_\tau^\beta \|\mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi_n(t), \chi_n(t)) - \mathcal{H}_\omega(t, \mathcal{I}^\sigma\chi(t), \chi(t))\|\right)\right. \\
& \quad \left. + \frac{s^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\left(\mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi_n(s), \chi_n(s)) - \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\|)\right)\right)ds \\
& \quad + \mathcal{I}_\kappa^\alpha\left(\mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi_n(s), \chi_n(s)) - \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\|)\right) \\
& \quad + \frac{\kappa^\alpha}{T^\alpha}\mathcal{I}_T^\alpha\left(\mathcal{I}_\tau^\beta(\|\mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi_n(s), \chi_n(s)) - \mathcal{H}_\omega(s, \mathcal{I}^\sigma\chi(s), \chi(s))\|))\right) \\
& \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Finally, Λ_2 is continuous.

Let us now prove that the set $\{\Lambda_2\chi(\kappa) : \chi \in \mathcal{B}_r(0,E)\}$ is relatively compact in X for all $\kappa \in [0,T]$. One can remark that $\{\Lambda_2\chi(\kappa) : \chi \in \mathcal{B}_r(0,E)\}$ is compact, fix $\kappa \in (0,T]$ and $\chi \in \mathcal{B}_r(0,E)$, and define the operator Λ_2^{ε}:

$$\begin{aligned}\Lambda_2^{\varepsilon}\chi(\kappa) = \int_0^{\kappa-\epsilon} &\mathcal{R}'(\kappa-s)\Big(\mathcal{I}_s^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\Big)\\ &-\frac{s^{\alpha}}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\Big)\Big)ds\\ &+\mathcal{I}_{\kappa-\epsilon}^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\Big)-\frac{(\kappa-\epsilon)^{\alpha}}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\Big),\ \omega \geq 0.\end{aligned}$$

Due to the complete continuity of $\mathcal{H}_\omega$, as stated in **(H3)**, we can conclude that for every $\epsilon > 0$ with $0 < \epsilon < \kappa$, the set

$$\Omega_\epsilon = \{\Lambda_2^{\epsilon}\chi(\kappa) : \chi \in \mathcal{B}_r(0,E)\},$$

is precompact in X. Moreover, for every $\chi \in \mathcal{B}_r(0,E)$, we have

$$\begin{aligned}&\|\Lambda_2\chi(\kappa)-\Lambda_2^{\epsilon}\chi(\kappa)\|\\ &\quad\leq \int_{\kappa-\epsilon}^{\kappa}\mathcal{R}'(\kappa-s)\Big(\mathcal{I}_s^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\|\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\|\Big)\\ &\qquad+\frac{s^{\alpha}}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\|\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\|\Big)\Big)ds\\ &\qquad+\frac{1}{\Gamma(\alpha)}\int_{\kappa-\epsilon}^{\kappa}(\kappa-\tau)^{\alpha-1}\Big(\mathcal{I}_\tau^{\beta}\|\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\|\Big)d\tau\\ &\qquad+\frac{\big(\kappa^{\alpha}-(\kappa-\epsilon)^{\alpha}\big)}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}(\|\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\|)\Big).\end{aligned}$$

This means that the precompact sets Ω_ϵ, $0 < \epsilon < \kappa$,are close in the set $\{\Lambda_2\chi(\kappa) : \chi \in \mathcal{B}_r(0,E)\}$. Hence, the set $\{\Lambda_2\chi(\kappa) : \chi \in \mathcal{B}_r(0,E)\}$ is precompact in X.

Next, our goal is to establish that $\Lambda_2(\mathcal{B}_r(0,E))$ is equicontinuous. The functions $\Lambda_2\chi, \chi \in \mathcal{B}_r(0,E)$ are equicontinuous at $\kappa = 0$; then, if $\kappa < \kappa + h \leq T, h > 0$, we have

$$\begin{aligned}&\|\Lambda_2\chi(\kappa+h)-\Lambda_2\chi(\kappa)\|\\ &\quad\leq \Big\|\int_0^{\kappa+h}\mathcal{R}'(\kappa-s)\Big(\mathcal{I}_s^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\Big)\\ &-\frac{s^{\alpha}}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\Big)\Big)d\kappa\\ &-\int_0^{\kappa}\mathcal{R}'(\kappa-s)\Big(\mathcal{I}_s^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\Big)\\ &-\frac{s^{\alpha}}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(t,\mathcal{I}^{\sigma}\chi(t),\chi(t))\Big)\Big)d\kappa\Big\|\\ &\qquad+\Big\|\mathcal{I}_{\kappa+h}^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\Big)-\mathcal{I}_\kappa^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\Big)\Big\|\\ &\qquad+\Big\|\frac{\big((\kappa+h)^{\alpha}-\kappa^{\alpha}\big)}{T^{\alpha}}\mathcal{I}_T^{\alpha}\Big(\mathcal{I}_\tau^{\beta}\mathcal{H}_\omega(s,\mathcal{I}^{\sigma}\chi(s),\chi(s))\Big)\Big\|.\end{aligned}$$

So, we obtain

$$\lim_{h\to 0}\|\Lambda_2\chi(\kappa+h)-\Lambda_2\chi(\kappa)\| = 0.$$

According to hypothesis (H3), the function $\mathcal{H}_\omega$ iscompletely continuous. This means that the set

$$\{\Lambda_2\chi(\kappa) : \chi \in \mathcal{B}_r(0,E)\},$$

is equicontinuous. Therefore, we have demonstrated that $\Lambda_2(\mathcal{B}_r(0,E))$ is relatively compact for $\kappa \in [0,T]$. By applying the Arzela–Ascoli theorem, we conclude that Λ_2 is a compact operator.

Consequently, based on the Krasnoseskii fixed-point theorem, there exists a fixed point $\chi \in C([0,T],X)$ such that $\Lambda\chi = \chi$, with $\Lambda = \Lambda_1 + \Lambda_2$. This fixed point represents a mild solution to Problem (1). $\square$

4. Particular Case $\mathcal{A} \equiv \lambda$

We consider the following problem:

$$\begin{cases} \mathcal{D}^{\beta}(\mathcal{D}^{\alpha} - \lambda)\chi(\kappa) = \mathcal{H}_{\omega}(\kappa, \mathcal{I}^{\sigma}(\chi(\kappa)), \chi(\kappa)),\ \kappa \in [0,T], \\ \chi(0) = g(\chi),\ \chi(T) = \lambda \mathcal{I}^{\alpha}_{T}(\chi), \end{cases} \tag{24}$$

where $\mathcal{D}^{\alpha}$, $\mathcal{D}^{\beta}$ are Caputo fractional derivatives, and $\mathcal{I}^{\sigma}$ is the Riemann–Liouville fractional integral, where $0 < \alpha < 1$, $0 < \beta, \sigma < 1$, $\lambda \in \mathbb{R}$. Here, $\mathcal{H}_{\omega}$ depends on a parameter $\omega \geq 0$, where $\mathcal{H}_{\omega} : [0,T] \times X^2 \to X$, $g : C(J,X) \to X$ are continuous functions, and X is a Banach space.

Problem (24) is equivalent to the following integral equation:

$$\begin{aligned} \chi(\kappa) = {} & \lambda \mathcal{I}^{\alpha}_{\kappa}(\chi(\tau)) + \mathcal{I}^{\alpha}_{\kappa}\Big(\mathcal{I}^{\beta}_{\tau}\mathcal{H}_{\omega}(s, \mathcal{I}^{\sigma}\chi(s), \chi(s))\Big) \\ & - \frac{\kappa^{\alpha}}{T^{\alpha}}\mathcal{I}^{\alpha}_{T}\Big(\mathcal{I}^{\beta}_{\tau}\mathcal{H}_{\omega}(s, \mathcal{I}^{\sigma}\chi(s), \chi(s))\Big) + g(\chi)\left(1 - \frac{\kappa^{\alpha}}{T^{\alpha}}\right),\ \kappa \in [0,T]. \end{aligned}$$

Corollary 1. *Under assumptions* ***(H2)****,* ***(H3)****,* ***(H4)*** *and* ***(H5)****, we can conclude the existence and uniqueness of a mild solution to Problem (24).*

5. Application

In this section, our focus is on investigating the existence and uniqueness of a mild solution for a differential system that involves Caputo derivatives.

Example 2. *We consider the following differential equation with a Caputo derivative in* $X = L^2([0,\pi])$:

$$\begin{cases} \mathcal{D}^{\frac{1}{2}}\left(\mathcal{D}^{\frac{2}{5}} + \frac{1}{2}\right)\chi(\kappa) = \dfrac{1}{\omega^2 + \omega}\mathcal{I}^{\frac{1}{3}}_{\kappa}(\chi(\kappa)), \\ \chi(0) = \dfrac{1}{\omega + 1}\cos(\chi), \\ \chi(1) = \mathcal{I}^{\frac{1}{2}}_{1}(-\frac{1}{2}\chi) = -\frac{1}{2\Gamma(\frac{1}{2})}\int_0^1 (\kappa - s)^{-\frac{1}{2}}\chi(s)ds, \end{cases} \tag{25}$$

where $\mathcal{A}\chi = \chi$ *and* $\mathcal{D}(\mathcal{A}) = X$*. Clearly, there exists a resolvent operator for this problem, with* $\|\mathcal{R}'(t)x\| \leq \theta\|x\|$.

For a sufficiently large value of ω*, we have*

$$\mathcal{H}_{\omega}(\kappa, \mathcal{I}^{\sigma}\chi(\kappa), \chi(\kappa)) = \frac{1}{\omega^2 + \omega}\left(\mathcal{I}^{\frac{1}{3}}_{\kappa}\chi(\kappa)\right),$$

and

$$g(\chi) = \frac{1}{\omega + 1}\cos(\chi),$$

so we obtain

$$\|\mathcal{H}_{\omega}(\kappa, \tau_1, s_1) - \mathcal{H}_{\omega}(\kappa, \tau_2, s_2)\| \leq \frac{1}{\omega^2 + \omega}(\|\tau_1 - \tau_2\| + \|s_1 - s_2\|) \textit{with } L_{\omega} = \frac{1}{\omega^2 + \omega},$$

and

$$\|g(\tau_1) - g(\tau_2)\| \leq \rho(\omega)\|\tau_1 - \tau_2\| \textit{ where } \rho(\omega) = \frac{1}{\omega + 1},$$

and $\lim_{\omega\to+\infty}\rho(\omega)=0$.

On the other hand, $\exists\omega^*\geq 0$ *such that* $\forall\omega\geq\omega^*$, *and we have*

$$\begin{aligned}&2(1+\|\phi_{\mathcal{A}}\|_{L^1})\left[\frac{2L_\omega}{\Gamma(\alpha+\beta+1)}\left(\frac{1}{\Gamma(\sigma+1)}+1\right)+\rho(\omega)\right]\\&=2(1+\theta)\left[\frac{2}{(\omega^2+\omega)\Gamma\left(\frac{9}{10}\right)}\left(\frac{1}{\Gamma(\frac{4}{3})}+1\right)+\frac{1}{\omega+1}\right]<1,\end{aligned}$$

Therefore, assumptions (H1)–(H5) are satisfied, so Problem (25) possesses a unique mild solution.

Example 3. *Consider the following partial differential equation with a Caputo derivative in the space* $X=L^2([0,\pi])$: *the problem for* $X=L^2([0,\pi])$ *is given by*

$$\begin{cases}\dfrac{\partial^{\frac{1}{2}}}{\partial\kappa^{\frac{1}{2}}}\left(\dfrac{\partial^{\frac{3}{4}}}{\partial\kappa^{\frac{3}{4}}}-\dfrac{\partial^2}{\partial\tau^2}\right)\chi(\kappa,\tau)=\dfrac{1}{\omega}\left(\sin(\kappa)\chi(\kappa,\tau)+\mathcal{I}_\kappa^{\frac{1}{2}}(\chi(\kappa,\tau))\right),\\ \chi(\kappa,0)=\chi(\kappa,\pi)=0,\ \kappa\in[0,1],\\ \chi(0,\tau)=\dfrac{1}{\omega+1}\sin(\chi),\\ \chi(1,\tau)=\mathcal{I}_1^{\frac{1}{2}}\left(\dfrac{\partial^2\chi}{\partial\tau^2}\right)=\frac{1}{\Gamma(\frac{1}{2})}\int_0^1(\kappa-s)^{-\frac{1}{2}}\dfrac{\partial^2\chi}{\partial\tau^2}(s,\tau)ds,\ \tau\in J=[0,\pi],\end{cases}\tag{26}$$

where $\alpha=\frac{1}{2},\beta=\frac{3}{4},\sigma=\frac{1}{2}$, *and* $\omega>0$ *is sufficiently large.*

Let $\mathcal{A}\chi=\chi''$ *with the domain*

$$\mathcal{D}(\mathcal{A})=\{\chi\in X,\chi''\in X,\ \chi(0)=\chi(\pi)=0\}.$$

Due to [15,39], the integral equation

$$\chi(\kappa)=\frac{1}{\Gamma(\alpha)}\int_0^\kappa(\kappa-s)^{\alpha-1}\mathcal{A}\chi(s)ds,\kappa\geq 0,\tag{27}$$

has an associated resolvent operator $(\mathcal{R}(t))_{t\geq 0}$ *on X, and there exists a constant* $\eta>0$ *such that*

$$\|\mathcal{R}'(t)x\|\leq\eta\|x\|,\ \textit{for all } t>0,\ \textit{for } x\in\mathcal{D}(\mathcal{A}).$$

*Therefore, we can confirm that assumptions **(H1)** and **(H2)** are satisfied. For a sufficiently large value of* ω, *we have*

$$\mathcal{H}_\omega(\kappa,\mathcal{I}^\sigma\chi(\kappa),\chi(\kappa))=\frac{1}{\omega}\left(\sin(\kappa)\chi(\kappa,\tau)+\mathcal{I}_\kappa^{\frac{1}{2}}\chi(\kappa,\tau)\right),$$

and

$$g(\chi)=\frac{1}{\omega+1}\sin(\chi),$$

so, we obtain

$$\|\mathcal{H}_\omega(\kappa,\tau_1,s_1)-\mathcal{H}_\omega(\kappa,\tau_2,s_2)\|\leq\frac{1}{\omega}(\|\tau_1-\tau_2\|+\|s_1-s_2\|)\textit{with } L_\omega=\frac{1}{\omega},$$

and

$$\|g(\tau_1)-g(\tau_2)\|\leq\rho(\omega)\|\tau_1-\tau_2\|\ \textit{with } \rho(\omega)=\frac{1}{\omega+1},$$

and $\lim_{\omega\to+\infty}\rho(\omega)=0$.

Then, we can conclude that both **(H3)** *and* **(H4)** *are satisfied. On the other hand, we have*

$$2(1+\|\phi_{\mathcal{A}}\|_{L^1})\left[\frac{2L_\omega}{\Gamma(\alpha+\beta+1)}\left(\frac{1}{\Gamma(\sigma+1)}+1\right)+\rho(\omega)\right]$$
$$=2(1+\|\phi_{\mathcal{A}}\|_{L^1})\left[\frac{2}{\omega\Gamma(\alpha+\beta+1)}\left(\frac{1}{\Gamma(\sigma+1)}+1\right)+\frac{1}{\omega+1}\right],$$

and

$$2(1+\|\phi_{\mathcal{A}}\|_{L^1})\left[\frac{2}{\omega\Gamma\left(\frac{5}{4}+1\right)}\left(\frac{1}{\Gamma\left(\frac{1}{2}+1\right)}+1\right)+\frac{1}{\omega+1}\right]$$
$$<2(1+\|\phi_{\mathcal{A}}\|_{L^1})\left[\frac{2}{\omega\Gamma\left(\frac{5}{4}+1\right)}\left(\frac{1}{\Gamma\left(\frac{1}{2}+1\right)}+1\right)+\frac{1}{\omega}\right].$$

So,

$$2(1+\eta)\left[\frac{2}{\omega\times 1.13}\left(\frac{1}{0.88}+1\right)+\frac{1}{\omega}\right]<1,$$

which is equivalent to

$$(1+\eta)\left[\frac{4}{1.13}\left(\frac{1}{0.88}+1\right)+2\right]<\omega$$

This means $\omega > 9.54\times(1+\eta)$.

Then, there exists $\exists\omega^* = 9.54(1+\eta) > 0$ *such that, for all* $\omega > \omega^*$,

$$\Upsilon=2(1+\|\phi_{\mathcal{A}}\|_{L^1})\left[\frac{2L_\omega}{\Gamma(\alpha+\beta+1)}\left(\frac{1}{\Gamma(\sigma+1)}+1\right)+\rho(\omega)\right]<1.$$

Therefore, for any $\omega > \omega^* = 9.54\times(1+\eta)$, *Problem (26) possesses a unique mild solution.*

6. Conclusions

In this study, we have extended the concept of sequential fractional differential equations by introducing an operator coefficient, thus creating what we refer to as an abstract sequential fractional differential equation. We have examined the uniqueness and existence of mild solutions to such abstract sequential fractional differential equations with nonlocal boundary conditions. Our investigation utilizes the Caputo fractional derivative and Riemann–Liouville fractional integral operators, with a particular focus on the role of resolvent operators. To establish uniqueness, we apply the Banach contraction principle, while, for existence, we utilize the Krasnoseskii fixed-point theorem. We also provide an application of our results to a partial differential equation to demonstrate their applicability to practical problems. In our future research, we will concentrate on investigating the Ulam–Hyers and Ulam–Hyers–Rassias stability of similar problems using the approach of resolvent operators.

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