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Editorial

Fractional Calculus and Hypergeometric Functions in Complex Analysis

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1. Introduction

Fractional calculus has had a powerful impact on recent research, with many applications in different branches of science and engineering. Various branches of mathematics are also influenced by fractional calculus. Articles in the literature [1,2] discuss the history of fractional calculus and cite its many scientific and engineering applications, successfully highlighting the importance of this subject. Fractional operators are essential to the study of fractional calculus. For investigations utilizing fractional calculus, fractional operators are crucial resources. A brief history of fractional calculus operators is given in [3] and is further developed in [4]. Applications of fractional operators in complex analysis research are comprehensive, and interesting new results have been obtained in studies involving univalent functions theory, a topic which is also covered in [1].

This Special Issue aims to gather new research outcomes combining this prolific tool with another that generates exciting results when integrated into studies: hypergeometric functions.

The study of hypergeometric functions dates back 200 years. They appear in the works of Euler, Gauss, Riemann, and Kummer. Interest in hypergeometric functions has grown in the last few decades due to their applications in a large variety of scientific domains and many areas of mathematics. Hypergeometric functions are linked to the theory of univalent functions by L. de Branges' proof of Bieberbach's conjecture, published in 1985 [5], which uses the generalized hypergeometric function. After this connection was established, hypergeometric functions were studied intensely using geometric function theory.

Quantum calculus is also involved in studies alongside fractional calculus tools and different hypergeometric functions, as is nicely highlighted in [6].

This Special Issue compiles articles from researchers interested in any of these topics or a combination of them and their applications in different areas concerning complex analysis.

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2. Overview of the Published Papers

After a thorough review procedure, 12 papers were selected for publishing in this Special Issue.

Najla M. Alarifi and Rabha W. Ibrahim (contribution 1) investigate the geometric properties of the generalized Prabhakar fractional differential operator in the open unit disk by using the concept of q–fractional calculus. The generalized operator is inserted in a special class of analytic functions. By using the methods of differential subordination and superordination theory, numerous fractional differential inequalities are proven. Additionally, this contribution investigates the potential application of these methods in the solution of special kinds of q–fractional differential equations.

The research presented by the authors Lei Shi, Muhammad Arif, Javed Iqbal, Khalil Ullah, and Syed Muhammad Ghufran (contribution 2) concerns the study of logarithmic-related problems of a certain subclass of univalent functions. A subclass of starlike functions connected with exponential mapping is introduced, and sharp estimates of the second Hankel determinant with the logarithmic coefficient as the entry are obtained for this class.

The well-known parametric formulas for initial coefficients in the Carathéodory class of functions serve as the methodological foundation for the proof. The authors discover that the bounds for the coefficients of a function and its inverse function can be obtained by transferring the logarithmic coefficients of functions. The results on Hankel determinants with logarithmic coefficients seem to be quite important, since bounds on coefficients of the inverse function are often a more challenging task to calculate. Since the exponential function belongs to a special class of hypergeometric functions, this work can serve as an inspiration for further research on univalent functions that are subordinated to a more general class.

The results presented by Georgia Irina Oros, Gheorghe Oros, and Shigeyoshi Owa (contribution 3) arise from a study regarding fractional calculus combined with the classical theory of differential subordination established for analytic complex valued functions. A new operator is introduced by applying the Libera integral operator and fractional integral of order λ for analytic functions. Many subordination properties are obtained for this newly defined operator by using famous lemmas proved by important scientists concerned with geometric function theory, such as Eenigenburg, Hallenbeck, Miller, Mocanu, Nunokawa, Reade, Ruscheweyh, and Suffridge. Results regarding strong starlikeness and convexity of order λ are also discussed, and an example shows how the outcome of this research can be applied. The operator defined in this work can be applied to the definition of new subclasses of analytic functions with certain geometric properties given by the characteristics of this operator that are already proven in this paper.

In the next paper (contribution 4), Muhammad Bilal Khan, Adriana Cătaş, Najla Aloraini, and Mohamed S. Soliman introduce left and right exponential trigonometric convex interval-valued mappings and review some of their important characteristics. The Hermite–Hadamard inequality for interval-valued functions is proven by utilizing fractional integrals with exponential kernels. Moreover, the idea of left and right exponential trigonometric convex interval-valued mappings is applied to show various findings for midpoint and Pachpatte-type inequalities. The authors also show that the results provided in this paper are expansions of several of the results that have already been demonstrated in prior publications. The suggested research following from this work generates variants that are applicable for conducting in-depth analyses of fractal theory, optimization, and research challenges in several practical domains, such as computer science, quantum mechanics, and quantum physics.

The investigation conducted by Mohammad Faisal Khan and Mohammed Abaoud (contribution 5) examines a new subclass of generalized bi-subordinate functions of complex order γ connected to the q-difference operator. The upper bounds for generalized bi-subordinate functions of complex order γ are obtained by using the Faber polynomial expansion technique. Coefficient bounds and the Fekete–Szegő problem are considered for functions in the newly defined class. The Ruscheweyh q-differential operator along with the Faber polynomial method are used to discuss the applications of the main results. The authors suggest that the method presented in this paper could be applied to define a number of new subclasses of meromorphic, multivalent, and harmonic functions and can be used to investigate a number of new properties of these classes.

The authors Sadia Riaz, Timilehin Gideon Shaba, Qin Xin, Fairouz Tchier, Bilal Khan, and Sarfraz Nawaz Malik investigate bi-univalent functions and Euler polynomials in their study (contribution 6). The Fekete–Szegő problem is solved by the authors, and bound estimates for the coefficients and an upper bound estimate for the second Hankel determinant are given for the new class of bi-univalent functions satisfying a certain subordination and involving Euler polynomials. The authors believe that their results could be extended for a class of certain q-starlike functions.

A new computational technique for solving some physics problems involving fractional-order differential equations, including the famous Bagley–Torvik method, is given by Hari Mohan Srivastava, Waleed Adel, Mohammad Izadi, and Adel A. El-Sayed (contribution 7). A collocation technique involving a new operational matrix that utilizes the Liouville–Caputo operator of differentiation and Morgan–Voyce polynomials is adapted in combination with the Tau spectral method. The differentiation matrix of fractional order that is used to convert the problem and its conditions into an algebraic system of equations with unknown coefficients is first presented. Then, the matrix is used to find the solutions to the proposed models. An error analysis for the method is proven to verify the convergence of the acquired solutions. To test the effectiveness of the proposed technique, several examples are simulated using the presented technique, and these results are compared with other techniques from the literature. In addition, the computational time is computed and tabulated to ensure the efficacy and robustness of the method. The outcomes of the numerical examples support the theoretical results and show the accuracy and applicability of the presented approach.

The authors Mohammed Z. Alqarni, Ahmed Bakhet, and Mohamed Abdalla (contribution 8) establish a generalization of the fractional kinetic equation using the generalized incomplete Wright hypergeometric function. This new generalization can be used to compute the change in chemical composition in stars such as the Sun. The pathway-type transform technique is then used to investigate the solutions to a fractional kinetic equation with specific fractional transforms. Furthermore, exceptional cases of the outcomes are discussed and graphically illustrated using MATLAB software. This work provides a thorough overview for further investigation into these topics in order to gain a better understanding of their implications and applications.

The next article (contribution 9) investigates the geometric properties of analytic functions using q-analogues of differential and integral operators. The authors Suha B. Al-Shaikh, Ahmad A. Abubaker, Khaled Matarneh, and Mohammad Faisal Khan define the q-analogues of a differential operator by using the basic idea of q-calculus and the definition of convolution. Using the newly constructed operator, the q-analogues of two new integral operators are established. Further, by employing these operators, new subclasses of the q-starlike and q-convex functions are defined. Sufficient conditions for the functions to belong to the newly defined classes are investigated, and certain subordination findings for the q-analogue differential operator are given. Certain novel geometric characteristics of the q-analogues of the integral operators in these classes are also obtained.

A comprehensive investigation to identify the uses of the Sălăgean q-differential operator for meromorphic multivalent functions is conducted by Isra Al-Shbeil, Jianhua Gong, Samrat Ray, Shahid Khan, Nazar Khan, and Hala Alaqad (contribution 10). In their paper, they extend the idea of the q-analogues of the Sălăgean differential operator for meromorphic multivalent functions using the fundamental ideas of q-calculus. With the help of this operator, the family of Janowski functions is extended by adding two new subclasses of meromorphic q-starlike and meromorphic multivalent q-starlike functions. The radii of starlikeness, partial sums, distortion theorems, and coefficient estimates are given for the new subclasses under investigation. The technique and ideas of this paper may stimulate further research in the theory of multivalent meromorphic functions, and additional generalized classes of meromorphic functions can be defined and investigated.

The next article (contribution 11) introduces three general double-series identities using Whipple transformations for terminating generalized hypergeometric ${}_4F_3$ and ${}_5F_4$ functions. By employing the left-sided Riemann–Liouville fractional integral on these identities, the authors Mohd Idris Qureshi, Tafaz Ul Rahman Shah, Junesang Choi, and Aarif Hussain Bhat show the ability to derive additional identities of the same nature successively. This research further presents various new transformation formulae, such as Bailey's quadratic transformation formula, the Clausen reduction formula, the Gauss quadratic transformation formula, the Karlsson reduction formula, the Orr reduction formula, and the Whipple quadratic transformation formula. The authors anticipate that these transformation and summation formulas, as well as those deducible from the same steps, will have applications in diverse fields, such as mathematical physics, statistics, and engineering sciences.

The authors Adeel Ahmad, Jianhua Gong, Isra Al-Shbeil, Akhter Rasheed, Asad Ali, and Saqib Hussain (contribution 12) define a new generalized domain obtained based on the quotient of two analytic functions. The sharp upper bounds of the modulus of the coefficients a_2, a_3, and a_4 are investigated, and the sharp upper bounds for the modulus of the second-order and third-order Hankel determinants are estimated for the normalized analytic functions belonging to the newly defined class in the generalized domain. This work provides a direction to define more interesting generalized domains and to extend to new subclasses of starlike and convex functions by using quantum calculus.

3. Conclusions

A printed book bearing the same title is available that contains the 12 papers published in this Special Issue on "Fractional Calculus and Hypergeometric Functions in Complex Analysis". This project has resulted in the publication of articles covering a wide range of topics. Because of this, scholars studying the applications of fractional calculus and hypergeometric functions in complex analysis and related fields should find this Special Issue to be interesting. This Special Issue's sequel is named "Fractional Calculus, Quantum Calculus, and Special Functions in Complex Analysis". In order to learn more about the suggested themes and perhaps contribute to the success of this new initiative by submitting research outputs, scholars interested in the field are welcome to visit the Special Issue homepage.

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