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A Theory of Best Choice Selection through Objective Arguments Grounded in Linear Response Theory Concepts

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Abstract: In this study, we propose how to use objective arguments grounded in statistical mechanics concepts in order to obtain a single number, obtained after aggregation, which would allow for the ranking of "agents", "opinions", etc., all defined in a very broad sense. We aim toward any process which should a priori demand or lead to some consensus in order to attain the presumably best choice among many possibilities. In order to specify the framework, we discuss previous attempts, recalling trivial means of scores—weighted or not—Condorcet paradox, TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), etc. We demonstrate, through geometrical arguments on a toy example and with four criteria, that the pre-selected order of criteria in previous attempts makes a difference in the final result. However, it might be unjustified. Thus, we base our "best choice theory" on the linear response theory in statistical physics: we indicate that one should be calculating correlations functions between all possible choice evaluations, thereby avoiding an arbitrarily ordered set of criteria. We justify the point through an example with six possible criteria. Applications in many fields are suggested. Furthermore, two toy models, serving as practical examples and illustrative arguments are discussed.

Keywords: best objective choice; complex systems; linear response theory; opinion formation; selection by ranking; sociophysics

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1. Introduction

Nowadays, statistical physics has found wide open research topics outside of its classical aims in economics and sociology nowadays [1–5]. Thus, consider the interplay between sociology and physics: sociophysics.

Forget Hobbes, Quetelet, Comte, Verhulst, and others there should be no need to point out, as an introduction or justification for this paper, that Galam is a pioneer of modern sociophysics [6,7] (see also the relevant papers [8–15] of interest for consideration in this paper).

One of Galam's goals was to set a framework, provide techniques, and search for conclusions on the dynamics of opinion formation in various societies. Galam's studies lead to the finding of the conditions for consensus (kind of an equilibrium), chaotic states, and intermediary complex phases in multi dimension diagrams. However, it is not quite apparent that his findings pertain to what people would refer to or call "the best choice".

But, what is the best choice? It should be admitted that what one would call the best choice is a highly relative and subjective state or concept. The answer contains ingredients from both personal ("selfish") and global ("self-effacing") points of view. Moreover, the dependence on exogenous and endogenous conditions is quite significant, but such considerations are left for political discussions elsewhere. We consider that the final states in Galam's studies or models, and subsequent studies by many, result from a too heavily weighted stochastic set of constraints or hypotheses. Opinion dynamics surely result from individual "votes" (meaning choices) due to herding or because of contrarians [9,12,16]. Nevertheless, do the dynamics imply a good choice? Or worse, is the choice (meaning vote) the best choice? That seems to be a crucial point, not only at election times in democracies, but also in choices like (we limited the references to a few papers) in the media [17], including music genrefication networks [18]; in economics [19], including regional studies [20] and drawdown market price sizes at speculative times [21], in academia, including scholarly journal rankings [22], research networks clustering [23], world universities ranking [24,25] or samba schools ranking [26,27]; and in sport [28–31].

More explicitly, in the academic domain, what is the best choice when hiring or promoting a colleague, when appointing a vice-chancellor, or when selecting teams for research grants? In the sport domain, the ranking of football teams or of cyclist racing teams is obtained through apparently objective numbers, but the rules can often (or even always) be debated and challenged [32,33]. The same remark holds for the Nobel Prize, the Pulitzer Prize, the Goncourt Prize, the Oscar or Cesar Awards which are given through highly subjective, not objective, criteria; not discounting facing a choice between roads going from X to Y, for going on holidays or to a restaurant, etc.? What is the best equipment or car or cell phone to buy? What is the best food, from a health point of view? All these questions mix subjective and objective criteria.

This boils down to the fundamental and practical question: what are the criteria needed for reaching the best choice? In other words, how should one conclusively rank a set of "things", "people" or "teams" in a constrained set of criteria?

This leads us to remember that the "final choice" leads to a somewhat paradoxical situation, the Condorcet paradox [34–36], for example. Recall that the Condorcet method [34–36] is a voting system that will always select the candidate that voters prefer over each other candidate, when comparing between them one at a time. Further considerations on comparing preferred choices lead to Arrow's incompatibility theorem [37]. Moreover, the order of criteria might lead to ambiguities [38].

The discussion, and the subsequent answers, should pertain to a comparison of the evaluation methods according to criteria [39,40]. Most of these are based on previous achievements, even though their forecasting value, not mentioning consistency for future achievements or impact, are far from certain.

This drastically annoying deduction seems to stem from the plethora of "parameters", i.e., possible criteria. Practically, one turns toward aggregation processes [41–43], going from multi dimensions toward a single number. This makes life complicated enough when one turns toward the modelling. Thus, one wishes to have some indubitable argument; often, that means having rigorous mathematical theorems. However, this is often hard to implement, in particular for laymen (or lay women, or lay others). Therefore, mathematical arguments might be by-passed through physics concepts which allow metaphors and analogies, like in modern statistical mechanics or like in Galam's view of social thought dynamics, for example, on networks.

Therefore, after outlining elements for discussion from information theory, arguments from geometry, and surely arguments on complex systems and sociophysics, we propose a powerful argument grounded in linear response theory (LRT): in LRT, the coefficients (magnetic susceptibility, transport coefficients, etc.), measured in the laboratory and estimated theoretically, are discussed and defined through the correlations between the (fluctuations of the) dependent variables. Thus, we suggest calculating all correlation functions implying the relevant variables in the sociophysics research topics of interest, in particular when searching for opinion formation, as due to Galam. This idea seems to be missing in earlier considerations. We claim that this idea is leading to a more objective hierarchy of values in opinion formation choice, i.e., topics.

Within this set of considerations, a study framework can be conceived together with applications, thereby leading to the following structural content of this study.

Section 2 contains a brief review of "old" and "modern" techniques for selecting and ranking agents or events, like the rank-size "laws". We discuss such attempts, recalling trivial "means of scores", be they weighted or not, the Condorcet paradox, etc. We recall preference aggregation techniques, like the maximum likelihood rule (MLR) (Section 2.1), and the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) (Section 2.2).

However, the aggregation problems have two different aspects: rank aggregation or score aggregation. These are briefly distinguished in Section 2.3.

In particular, in Section 2.4, we start from geometrical aspects for ranking scores, or measures, and for later aggregating multi-valued measures. Furthermore, we demonstrate through such geometrical arguments that the order of criteria might make a difference on the final result, but this might be unjustified. It is easily observed that, if only three criteria are used, the procedure leads to an undebatable result. We show on a set of four criteria for a toy model that the order is drastically relevant and influences the outcome after aggregation.

We base our best choice theory in Section 3, upon the linear response theory. We indicate that one should be calculating correlations functions between all possible choice evaluations. Appendix A recalls the fundamentals of a linear response theory in statistical physics.

We conclude with some emphasis, if it is needed, on the finding that the best ranking is obtained from a method based on rigorous arguments for obtaining a final score and ranking. We are aware that subjective arguments often influence the final decision. We suggest applications in a few domains in Section 4. Two examples can be found in Appendix B.

2. Modern Ideas and Methods

In this Section, we consider ideas and methods for the ranking and selecting of the best outcome due to inequalities between events and agents, also called "variables", thereby assuming that a hierarchy takes places and leads to the best choice [44].

Before recalling modern ideas, let us remind the reader that the selection process leads to some so-called rank-size (RS) displays or tables. Indeed, RS analysis is the basic way of measuring disorder in a population: the largest "size" gets the first rank, and a hierarchy, from which an ordering through inequality is deduced [44]. This is a hierarchical description. *In fine*, this leads to considering whether empirical laws follow patterns, thereby suggesting models.

The RS law derives from the analytical form presented by the variables, often ranked in descending order as a function of the (discrete) index i, giving a "rank" to each of the g_i, $i = 1, \dots, N$ variables. The cumulative law $\sum_i g_i$ leads to the cumulative concentration distribution curve (CCDC). The sum $\sum_i^N g_i$ over all the i elements can serve as a normalisation value. One immediately obtains the "normalised" CCDC . When the normalised CCDC goes over the 80% threshold, this defines the Pareto rank, r_P. The Pareto principle expects this rank r_P to be equal to $N/5$. Thus, the RS method is convenient, and better suitable for large populations, i.e., when the size or/and rank ranges can be large. This is rarely the case when only a few selection criteria are implied.

For completeness, nevertheless, let us mention that Marfels [45] distinguishes several types of concentration ratios according to weighting schemes and their structure, which can be discrete or cumulative [46]. Beside such ratios, inequality aspects are often discussed in order to touch upon socio-economic concerns [47].

Furthermore, one can recall that, when the goal is to find a compromise between the various rankings, the statistical median is thought to be the most appropriate solution [48,49].

However, in situations when decision making should be a way of compromising between conflicting decisions, the use of MLR, discussed in the following Section 2.1, is justified.

2.1. Maximum Likelihood Rule

Recall that a choice demands some ranking. This is usually done by combining ordered preference lists into a single consensus value, as in a reviewing procedure [49,50]. We outline the MLR, which is based directly on rankings and not on the scores.

In brief, methods of preference aggregation, such as the MLR, are based on the concept of pairwise preference notions [51], which are much used in economics and in opinion formation, alongside the simple majority voting rule, without a discussion of the scores. It is also known as the Kemeny [52] rule (or the Kemeny–Young method [48]). In order to go beyond the limit of the method, one introduces a variant, taking into account the behavioural argument of "blindness to small changes" (BSC) [53].

Mathematically, an MLR ordering is defined as one that minimises the total number of discrepancies among all the reviewers in their *pairwise preferences* between all options. It can be viewed as a voting scheme that determines not just a single chosen winner, but an entire ordered list. Therefore, the MLR ordering generally satisfies the most possible reviewers regarding their stated rankings of options. The ordering does not use whatever information about how much higher a candidate is ranked over another, but only a relative ordering (it is worth pointing out that MLR is a Condorcet method).

Practically, one can say that the method counts the *pairwise preferences* and applies the majority rule over each of them.

Examples abound on applying this rule and finding "solutions" [54]. One short illustration is given in Appendix B.

The second step of the method is counting the number of times where one candidate is ranked over another. In doing so, the MLR emphasises that the first-ranked choice is preferable compared to all other options in individual pairwise comparison. Similarly, the MLR second ranked choice would be be preferred to all other options (except the first ranked), and so on.

In order to complete this Section, we may consider that the MLR implies that candidates with the same score are considered to belong to the same "indifference set". If this occurs, one decides to list the "candidates" (specifying the "agents" or "events") one after the other, e.g., in alphabetical order, in such a set. One may consider other ways to manage equal scoring, but there is no need to foresee any special treatment here to take care of ex aequo positions in the ranking. Nevertheless, in most evaluation procedures, a linear order is requested for the final ranking; the most unwanted point is when two "agents" are ex aequo on the first rank. In general, in order to achieve such a goal, or resolve such a dilemma, extra information is a posteriori added for discriminating equally scored candidates. This might be unjustified but this "solution" is left for more suitable considerations and is beyond outside of the scope of the present study.

2.2. Technique for Order Preference by Similarity to Ideal Solution

The TOPSIS, was originally developed by Hwang and Yoon [55], with further pertinent developments, e.g., in Refs. [56–60], is a method of compensatory aggregation that compares a set of alternatives by identifying weights for each criterion, normalising scores for each criterion and calculating the geometric distance between each alternative and the ideal alternative, which is the best score in each criterion.

TOPSIS works attractively in various areas; see, e.g, [61].

2.3. Score Rather than Rank Aggregation

However, some fundamental emphases between processes in the preferential ordering problems must be made through distinguishing rank aggregation from score aggregation. Often, the ranks are not known, but are deduced from scores. However, as pointed out, one does not always know how the scores are obtained (recall the number of stars in the

Michelin guide or the Gault and Millau scores for restaurants, or the ranking order on Trip Advisor). In such cases, scores are incompatible and sometimes incomprehensible; ranking only makes some sense, allowing for ex aequos.

However, if scores are known from a reliable set of measures and for a given set of criteria, some objective construction can follow, leading to some meaningful score aggregation. A scoring function can be defined as $f^{(X_j)}(s_i)$, with $s_1, s_2, s_3, \ldots, s_m$, being the scores on each i-th criterion, for a given "agent" X_j, with $j = 1, \ldots, n$.

The aggregation results, from the set of n "events", "agents" and the m scoring criteria, sorted, for example, in decreasing order, lead to finding the top-k "candidates" according to the scoring function, imposing $f([s_i]) < f([s_i'])$. The objective is straightforward: to compute the top-k "agents" with the minimum cost. While the target is apparent, the methods can be quite different, depending on the choice of f. The very first problem stems from the sorting out algorithm. Let us point to the Fagin algorithm [62], to the medrank algorithm [63] and to the threshold algorithm [64]. For some completeness, another method is the Borda count [65]: for each ranking, one assigns a score D equal to the number of objects it defeats. The total weight of D is the number of points it accumulates from all rankings.

However, these rank or score ranking methods can be challenged because they are missing a key ingredient, i.e., the selection order of the criteria.

2.4. On the Sequence of Criteria: A Geometrical Perspective

In brief, many (if not all) earlier studies ignore a crucial factor: the sequence of criteria. It can be pointed out and straightforwardly illustrated that the ascertainment of criteria has an effect on ranking values; this is common knowledge to anyone having participated in surveys [66,67] and/or selection processes.

Suppose that there are three criteria giving a numerical value (a, b and c, respectively) for the agent or event. One can define a coordinate system with three axes stemming from some origin O in equivalent directions (such that the angle between each axis is, therefore, $2\pi/3$) and plot the values on each "criterion axis". Next, one way to aggregate the three values is to consider them as three sides of a triangle, and calculate the triangle area, which becomes the "score".

In this case, one may recall that it would be necessary, in order to find the area of a triangle with three sides, that one uses the Heron's formula: indeed, the area S of a triangle with three known sides, a, b and c, is calculated from

$$S = \sqrt{s(s-a)(s-b)(s-c)}\,, \tag{1}$$

where s is the semi-perimeter of the triangle, i.e., $s = (a+b+c)/2$.

On the other hand, knowing two sides b and c around an angle A, the formula to calculate the area of a triangle is given by $(1/2)\,b\,c \cdot \sin A$. Obviously,

$$S = b\,c/2 \qquad \text{if} \qquad A = \pi/2, \tag{2}$$

while

$$S = (\sqrt{3}/4)\,b\,c \qquad \text{if} \qquad A = 2\pi/3 \qquad (\text{or} \;\; \pi/3). \tag{3}$$

In the latter case, this means that whatever the order of criteria, the total area, i.e., the sum of the three triangles areas, is invariant and equal to $S = (\sqrt{3}/4)\,(b\,c + a\,b + a\,c)$.

However, the matter is different when there are more than three criteria or sustaining axes. Consider the case of four criteria. Let the axes form a coordinate system with four axes in symmetric directions. The permutation of axes leads to six different polygons. A toy example, with $a = 1$, $b = 2$, $c = 3$ and $d = 4$, can be seen in Figure 1. The four inner triangles in each polygon are rectangular triangles for which each area is easily obtained (the total area is given in Figure 1 in arbitrary units); it can be well noticed that, according to symmetry, only three different areas are relevant. Therefore, this leads to three different sizes if the area is considered to be the aggregated number for ranking the agents or events.

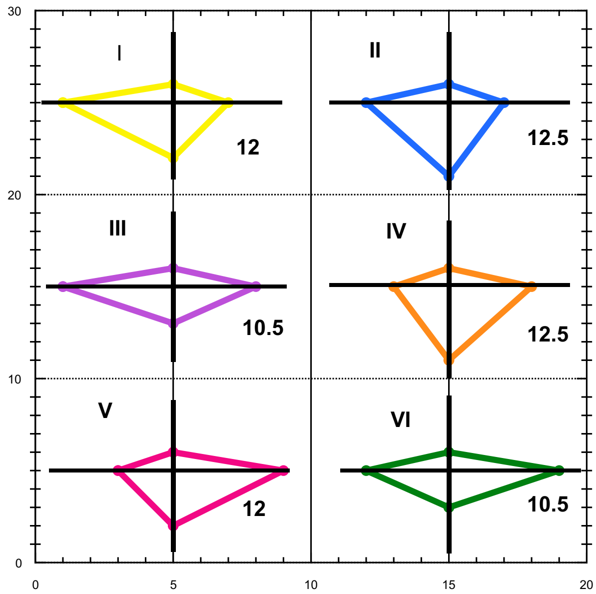

Figure 1. Demonstration that, starting from four criteria, the shape of the polygon based on the variable values leads to different areas. The toy side length values are $a = 1$, $b = 2$, $c = 3$ and $d = 4$ in arbitrary units), leading to six possible polygons, but three different sizes as indicated. See text for details.

When the number of criteria increases, from which many more polygons can be drawn, the area of such polygons can always be decomposed into a number of triangles, for which each area can be calculated using Equation (1), remembering that, using the length of two sides (a and b) of the known angle (C) between them, in any triangle, one can easily calculate the third side length of the triangle through the Al-Kashi formula, i.e.,

$$c^2 \;=\; a^2 \;+\; b^2 - 2\,a\,b \cdot \cos\; A\,. \tag{4}$$

3. The Problem and Its Solution

Thus, the way to proceed is as follows. First, take a survey with as many possible criteria that are needed for a population, whatever its size. One may suppose, without loss of generality, that the choice is based on Likert scales, whatever the useful range $[0,5]$, $[0,7]$, $\ldots$, $[0,20]$.

Notice that it could be of interest to consider the mean and standard deviation of the data distribution on each axis, such that one can obtain a possible "universal comparison" of final rankings and hierarchies. If the statistical characteristics make sense, the measured values could be normalised or, at least, constrained to vary in an identical range. One could assume that all the values involved in the decision process belong to $[0,1]$. The value 0 is achieved in the worst case for the considered parameter (bad realisation) and, accordingly, 1 is taken in the "best choice" case. Nevertheless, such a scaling is not a fundamental request. Although of interest, this normalisation aspect is not discussed further here and is left for further investigation.

Then, one reports the values on equidistant axes. These are disposed in a regular star-shaped network with a central node, a common starting point and identical angles between consecutive axes; the angle value actually depends on the number of axes (criteria). The non-common nodes of the consecutive segments are connected. The resulting polygon has a number of sides identical to the number of evaluation parameters; the polygon is not necessarily regular and can be considered as being made of triangles. If N is the number of axes, the largest possible number of different polygons is $(N-1)!/2$. The number of different triangles is $N!$.

In this paper, let us focus on hexagons, since evaluation processes are quite often associated with six different parameters.

In Figure 2, we present an example for two hexagons out of sixty possible cases; let a, b, c, d, e, f (e.g., $\in [0,1]$) be the values of the six parameters. The surface of the resulting hexagon $\mathcal{S}$ can be straightforwardly computed. The evaluation score coincides with the value of $\mathcal{S}$, so that a high $\mathcal{S}$ means a high score. However, notice that $\mathcal{S}$ belongs to the range associated to the limiting cases $a = b = c = d = e = f = 0$ (worst result of the evaluation process, with $\mathcal{S} = 0$) and $a = b = c = d = e = f = 1$ (best result of the evaluation exercise, with $\mathcal{S} = 3\sqrt{3}/2$).

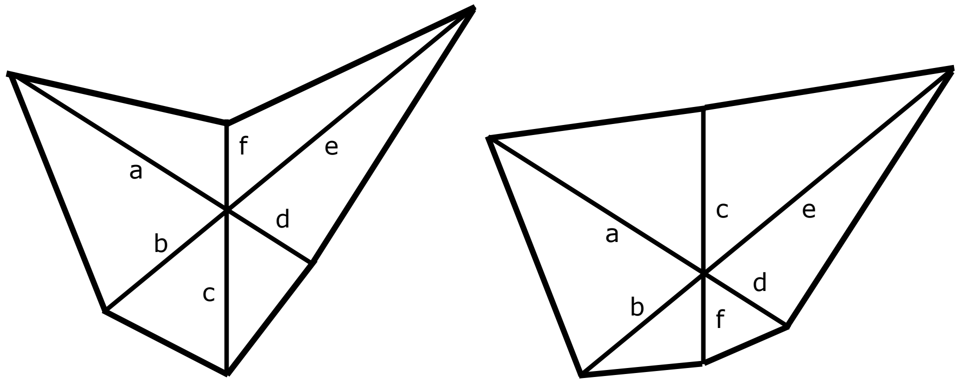

Figure 2. A toy case of six criteria for evaluation is presented with the parameter values a, b, c, d, e, and f with swapped axes c and f (**left** versus **right**). Notice that a different disposition in the relative order of the axes leads to a different area for the hexagon. See text for more details.

Certainly, the surface of the hexagons shown in Figure 2 depends on how the segments are disposed: the values of the six measures are always a, b, c, d, e and f, but the placement of c and f have been reverted in both displays.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the surface of the hexagons in Figure 2, left, and Figure 2, right, respectively. One has

$$\mathcal{S}_1 = \frac{\sin(\pi/3)}{2}(ab + bc + cd + de + ef + fa) = \frac{\sqrt{3}}{4}[(ab + bc + cd + de + ef + fa)],$$

$$\mathcal{S}_2 = \frac{\sin(\pi/3)}{2}(ab + bf + fd + de + ec + ca) = \frac{\sqrt{3}}{4}[ab + bf + fd + de + ec + ca];$$

In the case $b + d - e - a \neq 0$, one has that $\mathcal{S}_1 = \mathcal{S}_2$, if and only if $c = f$.

Such a remark leads to the questionability of the evaluation exercise through polygons. Indeed, the resulting polygons, differently shaped according to the disposition of the axes, have different surfaces. This means that the evaluation is strongly dependent on how selection parameters are included in the graphical representation of the problem, as we again stress.

To overcome such an inconsistency, and in order to obtain a more indubitable scoring, we propose taking the average $\bar{\mathcal{S}}(a, b, c, d, e, f)$ of all the possible polygon surfaces, thus over all the possible dispositions of the axes, here corresponding to six evaluation criteria:

$$\bar{\mathcal{S}}(a, b, c, d, e, f) = \frac{\sin(\pi/3)}{5!} \sum_{\mathbf{H}(a,b,c,d,e,f)} (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_1), \quad (5)$$

where

$$\mathbf{H}(a, b, c, d, e, f) \equiv (x_1, \ldots, x_6) \in \{a, b, c, d, e, f\} \quad (6)$$

with $x_i \neq x_j$ for each $i \neq j$.

By construction, $0 \leq \bar{S}(a,b,c,d,e,f) \leq 3\sqrt{3}/2$. Such a mean surface measure achieves the intended job effectively, i.e., providing a consistent score of the evaluated object whose six parameters (criteria) take values a,b,c,d,e,f.

This is consistent with the measure concept, resulting in calculating the correlation between fluctuations in statistical physics, as can be found in linear response theory: see Appendix A. Examples of applications are given in Appendix B.

4. Conclusions

To summarize, the present study is devoted to resolve the contradictory situation when searching for an objective ranking procedure and subsequently observing some hierarchy through a set of criteria for a given population. This sort of scientometric practice is tied to considerations found in geometry and statistical physics, complementing aspects of research on opinion dynamics found in Galam's numerous studies.

Observe the technical and somewhat philosophical frames in the present ranking study. Decisions often concern rather small sub-systems, like the funding of a project, the evaluation of a research group, or that of an individual at his or her hiring or promotion. Statistical laws cannot be immediately applied for such cases in which, due to the (very finite) size of the system, fluctuations which are not necessarily due to stochastic causes play a crucial role. Indeed, recall that scientific work is part of a social system; its actors are human beings. Moreover, statistics are like snapshots and do not often allow for predicting the future sufficiently well. In practice, indeed, any newly introduced criterion or "measure" will be met by the capacity of humans to interact and make decisions.

Without going back to the Bible and raising the question about who is the just and who will be saved (certainly, according to a single judge), one may, within modern scientific reasoning on choice, admit that when several partners and judges are implied, the resulting choice (solution) ends up into a difficulty pointed out by Arrow [37]. He considered the preference aggregation problem, that is the problem of passing from a set of known individual preferences to a pattern of social decision making. Arrow's today oft-quoted theorem, while being variously worded, has nevertheless shown that the difficulties met in the building process of preference aggregation are highly general. The theorem implies that no rank order voting system can convert the ranked preferences of individuals into a community-wide (complete and transitive) ranking, besides also meeting a specific set of natural criteria.

Why indeed is it impossible to reach a choice? Commonly, one constructs several filters and a priori decides on the order of their applications, like in the case of a decision tree (DT). That leads to a discussion on ranking the filters rather than the candidates.

Indeed, in the DT scheme, due to the order of filtering criteria, the final choice is significantly biased. The order of filters is often adapted a priori in order to select the final choice, e.g., maintaining a candidate, or candidates, in competition with others during the selection process, for hypocritical, political or other reasons.

In opinionology, the selection process consists in projecting from a multidimensional space onto various planes, and finally finding the intersection of the distribution. Recall that in physics, it is like projecting on various "external field axes" or making a scalar product. There might not be any "solution". This may further mean that the set (of "candidates", "events", etc.) to be ordered is either not to be ranked or that the filters are not appropriate.

Thus, we propose, starting from Galam's and others' considerations on the dynamics of choices, a procedure for obtaining the "best choice" through an objective statistical physics method.

We would like make a remark at this stage. One may say that our method is not better or worse than another, but is merely another type of classification; for example, it will not be applied by the Hollywood Academy to award Oscar Prizes. Let us notice that we are not using the word "better". We emphasise that our method is more rigorous, contains less arbitrariness and is based (through an analogy) on major statistical physics concepts. Certainly, we might regret that our method might not be applied in Los Angeles. Actually, we admit that we are not aware of all the criteria used by the Academy in order to award Oscars. The same holds true in other branches of opinion formation. In a decision process, the final ranking might be due to hidden or specifically weighted criteria; idem for the final choice. Our method does not apply outside the objective world. We do not consider subjective criteria.

Surely, the situation might not be closed, since the choice of criteria is left for many discussions between agents, be they surveyees, surveyors or other stakeholders.

Nevertheless, our argument, after outlining elements of reflexion from information theory, geometry and surely complex systems, as found in sociophysics considerations, proposes a strong argument for calculating choice values. It is grounded in linear response theory. In the latter, as it is recalled in Appendix A, the coefficients, that are measured in laboratories and estimated theoretically, are defined through the correlations between the dependent variables; thus, we suggest carrying out the same in sociophysics, i.e., calculating all correlation functions implying the relevant variables, thereby avoiding an arbitrarily ordered set of criteria. In particular, this could be useful and meaningful in the modelling of the opinion formation processes due to Galam. This methodology is missing in previous sociophysics studies. Thus, on a LRT basis, which is rather easily implemented, this should lead to a more objective hierarchy of values ahead of selection processes.

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Appendix A. Linear Response Theory

The LRT was independently invented by Green [68,69] and by Kubo [70]: it describes the coefficients relating the effect of a perturbation on a thermodynamic system in equilibrium. The LRT has given a general proof of the fluctuation–dissipation theorem, which states that the linear response of a given system to an external perturbation is expressed in terms of the correlations between the fluctuations of the relevant (with respect to the perturbation) variables characterizing the system in thermal equilibrium [71].

Kubo [70] considered the application of a magnetic field to an equilibrium system and demonstrated that the magnetic susceptibility can be defined through the average of the correlations between the magnetic moment density fluctuations. The LRT also well applies to the description of the electrical conductivity or the thermal conductivity [72,73], even if the system is characterised as being in a non-equilibrium state.

Consider a perturbation $B(t)$, e.g., a magnetic field, and search for the response $M(t)$ of the system, i.e., the magnetisation. Usually, one demands to obtain $M(t) = \chi B(t)$.

By analogy, in opinion formation and related rankings, one may consider that the "response" is going to be the final score. This is extracted through the correlations between the fluctuations of the values attributed to "candidates" (in a wide sense, usually called "agents'; however, they might not be humans) through the various criteria. The feature that criteria are introduced for later agent ranking is considered to induce "perturbations" in the original system. The external field is the exogenous decision of introducing criteria.

Subsequently, one may consider that the external (field) perturbation is applied at some time t_0 to some (for example) agents in a given population; the "population system" is thus moved away from its equilibrium and is characterised through a non-equilibrium ensemble average. Thus, this leads to the measured score of each agent as a result of interactions in response to the opinion field perturbation, as can be seen in most of Galam's models.

In practice, considering a weak interaction with some external field, one can obtain the resulting score by performing an expansion in powers of the perturbation [68–73]. The leading term in this expansion is independent of the field, but the next term describes the deviation from the equilibrium behaviour in terms of a linear dependence on the external perturbation through the correlations between the system fluctuations. The average score is the linear response function, i.e., the quantity that contains the (microscopic) information on the system and how it responds to the perturbing field.

Appendix B. Two Examples

Among the many possible examples among the various research fields outlined in Sections 1 and 4, we have selected two examples which are likely of interest to many readers and researchers in physics, particularly in sociophysics. The first example concerns the promotion of researchers. The second example pertains to the evaluation of football (soccer) players. The former stresses how to reach a final ranking from ranks obtained through various filters. Three methods, outlined in the main text, are compared. The latter example stresses that a meaningful ranking can be obtained from a final aggregation number based on measured values and analysed through correlations proposed in the main text.

Appendix B.1. Example 1: Researcher Promotion

Consider, as an illustration, that five researchers (A, B, C, D, E) are applying for a promotion. The relevant selection committee has decided to base its ranking of the value of the candidates on four criteria related to the impact factor (IF) of the journal where a paper has been published; from a practical point of view, let us note that the five candidates were required to submit their best ten papers , where the "best" being estimated due to the IF.

The committee calculates

- the IF mean of the best five papers: $\overline{B5}$;
- the IF mean of the worst five papers: $\overline{W5}$;
- the IF mean on the oldest five papers: $\overline{F5}$;
- the IF mean on the five most recent papers: $\overline{L5}$.

The timing of the publication is considered irrelevant at this stage. The resulting mean values and the corresponding candidate rank are given in Table A1 for each criterion.

Table A1. Profile of five candidates having published ten papers, listed in chronological order, with the candidate evaluation through various criteria, proposed by examiners. The best score value is underlined for each criterion. See text for details.

Candidate	$\overline{B5}$	Rank	$\overline{W5}$	Rank	$\overline{F5}$	Rank	$\overline{L5}$	Rank
A	$\underline{7.80}$	1	3.20	2	$\underline{7.80}$	1	3.20	4
B	7.00	4	2.20	3	2.20	4	7.00	2
C	7.60	2	$\underline{3.60}$	1	7.60	2	3.60	3
D	6.40	5	1.00	5	5.60	3	1.80	5
E	7.60	2	2.20	3	2.20	4	$\underline{7.60}$	1

Let the committee admit that (a) the ranks are more relevant than the scores and (b) the ranking of candidates is in descending order of the criteria scores. Then:

(i) It can immediately be seen that candidate A has the highest $\overline{B5}$ (=7.8) and $\overline{F5}$ (=7.8); author E has the maximum score on the last five ($\overline{L5}$) (=7.60), but author B does not reach any highest score, among these competitors or under any criterion. Let us also remark that, in this toy example, due to the nature of the evaluation, candidate D lays at the bottom on each ranking, no matter the specific criteria that are used. A complication arises in the need to rank the four others, since each of them is a winner for at least one committee member (or criterion): A and C even gain twice, but on different criteria certainly; B and E are twice ex aequos, but not near the top places. Thus, we have shown that different criteria lead to different rankings, but more so that the toy model implies that there is no obvious final choice as was, indeed, codified by Arrow's theorem. Moreover, there is no indubitable hierarchy along this simple aggregation process.

(ii) Next, consider the MLR method (Section 2.1) based on rankings, not on the scores, as given in Table A1. Recall that a MLR ordering is defined as one that minimises the total number of discrepancies among all the criteria in a pairwise preference scheme. The MLR ordering does not use any information about how much higher a candidate is ranked over another, but only a relative ordering. Practically, the method counts the pairwise preferences. The second step of the method consists in counting the number of times in which one candidate is ranked over another (from Table A1). In so doing, the MLR emphasises that the first-ranked choice wins against all other options in individual pairwise comparisons. Similarly, the MLR second-ranked choice would win against all other options (except the first-ranked), and so on. The present case outcome is reported in Table A2.

Table A2. Maximum likelihood rule (MLR) application: table reporting how many times, according to Table A1, the candidate in a row is ranked before the candidate in a column ex aequos. The parentheses indicate ex aequos, while the duplicated parentheses indicate two ex aequos.

Candidate	A	B	C	D	E
A	–	3	2	4	3
B	1	–	1	3	((0))
C	2	3	–	4	(2)
D	0	1	0	–	1
E	1	((2))	(1)	3	–

It can be observed from Table A2 that D has the lowest number of potential preferences for promotion, and hence is properly ranked as last. It can be remarked that the coalition of criteria formed only by $\overline{B5}$ and $\overline{L5}$ can be a decisive in the ranking of E before B. However, the rank relations between A and C are not so neat. Therefore, the selection committee would be faced, again, with the application of Arrow's theorem.

(iii) Finally, let us consider the newly proposed method. It boils down to calculating the surfaces of rectangular triangles and their subsequent averaging, i.e., here, for three types of polygons with four sides. Using Equation (2), one obtains the results displayed in Table A3. It looks that this ranking is justified. Thereafter, one may conclude that the proposed method is more justified and advantageous than classical ones.

Table A3. Ranking of five candidates according to the method proposed in this paper. The "Area" denotes the surface of each relevant 4-sided polygon, the "Total" denotes the sum of the area of such polygons, and the "Average" denotes the average area.

Candidate		Area (×2)		Total	Average	Rank
A	12	15	15	42	14	1
B	40	42	42	124	41.33	4
C	16	15	15	46	15.33	2
D	80	80	80	240	80	5
E	24	25	21	70	23.33	3

Appendix B.2. Example 2: Football (Soccer) Players

While writing this paper, we came across a related application we consider below. It concerns football players, ranking them on their "market value", "potential", salaries, age and many statistical "criteria" see [74] or, for teams, see [75]. Another example of "player value" can be found, for example in Ref. [76], where eight criteria are displayed. Notice that two axes—"yellow cards" and "red cards"—might both have a 0 value, reducing the display to a hexagonal pattern. Radar picture comparisons can be created [77].

In particular, e-players can form their own team, based on real players, considering six criteria, displayed on a regular hexagon and thereafter transformed into a colourful pattern, see, e.g., [78].

The six skill measures, obtained through several sub-criteria, are given in Table to the Table which those belong and abetter clarifyin there. Please consider Table A4.

Table A4. The six skill variables considered for measuring the "value" of a football player Eden Hazard, with their respective measure according to Ref. [78]. The data in the matrix corresponds to the area (divided by $[\sin(\pi/3)]/2$, see Equation (3)) of the fifteen possible triangles to be arranged in sixty different hexagons. Here, SHO stands for "shooting" and determines finishing skill and shot power, including penalty success; PAS stands for "passing" and denotes ability to successfully pass the ball with vision; DRI stands for "dribbling" and denotes ball control, agility and balance; DEF stands for "defending" and notes tackling and interceptions; PHY stands for "physicality" and notes strength, stamina and aggressiveness; PAC stands for "pace" and notes the speed and acceleration of the player.

Measure	Skill	SHO	PAS	DRI	DEF	PHY	PAC
77	SHO	-	6237	6545	2695	4774	6083
81	PAS	6237	-	6885	2835	5022	6399
85	DRI	6545	6885	-	2975	5270	6715
35	DEF	2695	2835	2975	-	2170	2765
62	PHY	4774	5022	5270	2170	-	4898
79	PAC	6083	6399	6715	2765	4898	-

Notice that it is not quite obvious how the "overall score" is obtained. Since only one hexagon is shown, it is certainly one for which the axes are a priori chosen, which does not conform to our recommendations about a tentative ranking (of football players) upon some unique value resulting from some aggregation process.

Thereafter, in order to avoid extra decimals, we measure the area of triangles and corresponding hexagons in $\sqrt{3}/4$ units; see Equation (3).

Consider one case in order to describe how the resulting aggregation player value is obtained, ahead of some selection process [78] according to which the "best overall" value of 82 of the player is obtained. However, the average of this playes skill values equals 69.83, as can be obtained from the values given in the Measure of Table A4. Furtrhermore, Table A4 reports the data corresponding to the area (divided by $[\sin(\pi/3)]/2$, see Equation (3)) of the fifteen possible triangles. For space saving, not all sixty relevant hexagons can be displayed, nor the entire list of their areas. For completeness, the statistical characteristics of the polygon areas' distribution are given in Table A5. The first line reports the characteristics for the distribution of triangle areas on the diagram displayed in Ref. [78]; this corresponds to at least six-axes diagram. The second line details the distribution of the 60-hexagon areas considered according to the present method. It is remarkable that the LRT result shows a considerably narrower distribution, thus more convincing statistics. The sign of the skewness is also in favour of the LRT method.

Table A5. Main statistical characteristics of the distributions of six-triangle areas forming the single hexagon [78] and of the areas each of sixty possible hexagons [78] along the six skill axes, following the linear response theory (LRT) method. See text for details. Here, "Min", "Max" and "Std Dev" stand for the minimum, maximum and standard deviation of the distributions, respectively.

Source	Size	Min	Max	Total Area	Mean	Std Dev	Skewness	Kurtosis
[78]	6	2170	6885	29,248	4874.7	1912.2	−0.4457	−1.4123
LRT	60	28,298	29,600	1,734,432	28,907	437.04	0.3076	−1.5345

Not considering here a comparison with other players, we should point out that such a comparison demands a final aggregation score. In order to have a universal rule, we suggest that the best approach is to measure the "player hexagon area(s)" with respect to the perfect player, i.e., the largest regular hexagon. For the case of Ref. [78] and the LRT case, one obtains 0.4875 and 0.4818, respectively.

As a conclusion of this Appendix, let us notice that we have first debated how to reach a hierarchical selection through three methods using numerical examples. This illustration is based on a toy case in which set of five candidates are applying for a promotion through a four-criteria selection process. Next, we presented the entire scheme, resulting into the aggregation score for a given individual examined through six values or criteria, according to two methods.

We have stressed the objective advantages of the LRT based method in both examples.

Therefore, one may conclude from both toy cases here that the usability and the advantage of the LRT methodology both are well supporting considerations in the sociophysics interdisciplinary field when searching for objective ranking, leading to justified choice. Let us emphasise that the main justification is the use of correlation functions, as in the LRT, i.e., a pertinent statistical physics theory basis.

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