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# Aging in Some Opinion Formation Models: A Comparative Study

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**Abstract:** Changes of mind can become less likely the longer an agent has adopted a given opinion state. This resilience or inertia to change has been called "aging". We perform a comparative study of the effects of aging on the critical behavior of two standard opinion models with pairwise interactions. One of them is the voter model, which is a two-state model with a dynamic that proceeds via social contagion; another is the so-called kinetic exchange model, which allows a third (neutral) state, and its formed opinion depends on the previous opinions of both interacting agents. Furthermore, in the noisy version of both models, random opinion changes are also allowed, regardless of the interactions. Due to aging, the probability of changing diminishes with the age, and to take this into account, we consider algebraic and exponential kernels. We investigate the situation where aging acts only on pairwise interactions. Analytical predictions for the critical curves of the order parameters are obtained for the opinion dynamics on a complete graph, in good agreement with agent-based simulations. For both models considered, the consensus is optimized via an intermediate value of the parameter that rules the rate of decrease of the aging factor.

**Keywords:** aging; opinion formation model; voter model; kinetic exchange model; social dynamics

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**Citation:** Llabrés, J.; Oliver-Bonafoux, S.; Anteneodo, C.; Toral, R. Aging in Some Opinion Formation Models: A Comparative Study. *Physics* **2024**, *6*, 515–528. https://doi.org/10.3390/physics6020034

Received: 21 December 2023
Revised: 6 February 2024
Accepted: 16 February 2024
Published: 8 April 2024



## 1. Introduction

As is known, the ideas and concepts coming from the statistical and nonlinear physics have been applied successfully to understand social phenomena [1,2]. The main goal is to explain "macroscopic" emergent behavior in terms of the "microscopic" individual's components in the same way that one derives the laws of macroscopic physical systems, e.g., the equation of state, from microscopic information such as the forces amongst particles. One lesson we have learnt from the modern development of statistical physics is that the emergent behavior, manifested through macroscopic phase transitions, is largely dependent on some details of the microscopic interactions and that there are some universal features that only depend on a few ingredients, such as dimensionality, symmetries, etc. It is in this spirit that the modeling of social systems has relied on the development of very simple agent-based models that take as starting points some stylized facts about the interactions amongst agents. Consider, for example, the subject of the evolution of cultures. In a seminal paper, Robert Axelrod asked: "If people tend to become more alike in their beliefs, attitudes, and behavior when they interact, why do not all such differences eventually disappear?" [3]. He then introduced a model whose basic assumption is that interactions amongst individuals lead to an augmentation in the similarity between them. A detailed analysis [4] shows that, depending on the initial diversity, there is a phase transition between an ordered monocultural state and a disordered multicultural state, demonstrating the apparent paradox of the emergence of global polarization through the mechanism of local convergence. It was then established that the dimensionality of the space of interactions was a relevant variable of the model [5]. Similar ideas appear in other areas. In the context of opinion formation, for example, one expects, on the one hand, that social influences reduce differences between individuals via social contagion, but, on the other, interactions can also produce differentiation via repulsive forces, associated with anticonformist "contrarian" agents that tend to deviate from the behavior adopted by the neighbors in the network of connections [6,7]. Another example is Galam's model for minority opinion spread [8,9] that, although based on local majority rule, can lead to the hostile minority views having an advantage.

Other diverse mechanisms can compete with the tendency toward uniformity promoted by social contagion. One is the "noisy" behavior, ruled by a "social temperature" parameter, associated with imperfect imitations due to random factors or independent choices, such that random changes of opinion interpreted as idiosyncratic behavior can occur. Standard models that study the effect of independence include the noisy voter model and the noisy kinetic exchange model. In the former, agents adopt opinions $\pm 1$, copying the opinion of a random contact. This model has been studied under a variety of different, but formally equivalent setups [10–17]. In the latter, the states $\pm 1$ and 0 can be adopted via a rule that takes into account the current opinion of both interacting contacts [18,19]. In both cases, the rule of social influence acts with a probability of $(1 - a)$ while otherwise, with a probability of $a$, a random change can occur. That is, these models rely on the assumption that opinion dynamics are mainly governed by mechanisms of imitation or social contagion that mold opinion formation through the interaction between individuals, but random factors are always present. In both cases, noise opposes the system's ability to reach a self-organized state with a winning opinion. Furthermore, both present a critical value $a_c$ below which order can be achieved, the main difference being that $a_c \to 0$ in the thermodynamic limit, when the number of agents tends to infinity, for the noisy voter, while $a_c$ remains finite in the kinetic exchange model in the same limit.

Another ingredient that has been proven to be relevant in the modeling of social systems is that of "aging". The effects of aging have been widely explored in several areas of research. In the field of computational biology [20–22], aging is understood as the increase of the mortality of a species as its population becomes older. In chemistry, aging appears when the properties of a material change over time without the influence of external forces but due to, for example, thermal degradation [23] or photo-oxidation [24]. In the context of social systems, aging is the concept that the larger resistance an agent offers to changing its state of opinion, the longer it has been holding the current state. Initially termed as "inertia" [25], it was shown that the slowing down of the microscopic dynamics induced via aging actually decreased the time needed to reach the macroscopically ordered state state of consensus. In this respect, and in alignment with previous studies [25–31], we consider that aging acts on the mechanism associated with social contagion.

In this paper, we compare the effects of aging on the two above-mentioned paradigmatic representatives of the class of noisy opinion models, the voter model and the kinetic exchange model, highlighting the similarities and differences amongst them. To this end, we consider an all-to-all interaction scheme in which every agent is connected to every other agent and develop an adiabatic approximation able to provide us with the phase diagram, including the location of the critical points separating regions of consensus from disordered regions. Afterward, we compare the location of the critical points for both models.

The rest of the paper is organized as follows: In Section 2, we introduce a general setup to study models of opinion dynamics under the presence of idiosyncratic, random changes of opinion, as well as the interaction between the agents modulated by an aging mechanism. In Section 3, we derive a mean-field-type approach for the noisy voter model under an adiabatic approximation, leaving the more technical details for Appendix A.1. In Section 3, we also compare the results of numerical simulations for two different functional forms for the aging probability, algebraic and exponential, focusing mainly on the magnetization, measuring the amount of order, or amount of consensus, in the system. In Section 4, after briefly reviewing the main results for the noisy kinetic exchange model, we compare the results for the magnetization and the dependence of the critical value of the noise intensity as a function of a parameter measuring the rate of decrease of the aging probability, for the voter and the kinetic exchange model. Finally, in Section 5, we end with some conclusions and an outlook for future work.

## 2. Models of Noisy Opinion Dynamics with Aging

In this study, we analyze two models that have been commonly used as prototypical examples in the study of the dynamics of opinion formation in social systems: the noisy voter model and the noisy kinetic exchange model. Both models consider a set of $N$ agents endowed with an internal state variable $s_i$, $i = 1, \cdots, N$, representing the possible positions of agent $i$ concerning a given topic. For the voter model, the internal variable can take two possible values $s_i \in \{-1, +1\}$, representing a position against or in favor of the topic. The kinetic exchange model introduces a third neutral state such that $s_i \in \{-1, 0, +1\}$. This internal variable evolves due to two different generic mechanisms that act stochastically: idiosyncratic changes and social influence. The difference between them is that the former randomly changes the state variable $s_i$ independently of the states of other agents, while the latter changes $s_i$ following a rule that depends on the state variables of another agent $s_j$. In the voter model, the social rule is that $s_i \to S(s_i, s_j) = s_j$, modeling the mechanism of imitation. In the kinetic exchange model, the social rule is $s_i \to S(s_i, s_j) = \text{sgn}[s_i + s_j]$, where $\text{sgn}[s]$ is the sign function. The rule is such that neutral agents can not modify the opinion of another agent, and agents with a well-defined opinion (either positive or negative) can convince a neutral agent or turn to the neutral state an agent with the opposite opinion. Let us mention that other rules for three-state models have been considered in the literature before [32–35]. In Ref. [32], it is shown that the inclusion of a third neutral state which individuals necessarily have to pass through increases the capability to reach consensus in the extreme values.

Previously, both models here considered have been studied under the influence of aging in the social mechanism [36,37]. In this paper, we generalize some of the previous studies and compare the effects that aging induces in both models.

To be precise, let us spell out in detail the rules of evolution for these models. Initially, we assign a random value to each state variable $s_i$ and set all internal times $\tau_i$ to zero. Then,

1. At each iteration, an agent $i$ is randomly selected.
2. With probability $(1 - a)$, the social rule is chosen: with probability $q(\tau_i)$, modeling the persistence or reaction of the individual $i$ to change as a function of its age $\tau_i$, a neighbor $j$ is randomly selected and the opinion of agent $i$ is modified according to the rule $s_i \to S(s_i, s_j)$.
3. Otherwise, with probability $a$, the idiosyncratic rule is chosen: the state $s_i$ is replaced by a randomly selected value among all possible opinion states.
4. Irrespective of the update mechanism actually used by agent $i$ (random change or pairwise interaction, with aging or not), its age $\tau_i$ is updated in the following way: if the state $s_i$ has changed, then age is reset, i.e., $\tau_i \to 0$; otherwise, age is incremented in one unit, i.e., $\tau_i \to \tau_i + 1$.

We restrict ourselves in this paper to the all-to-all connectivity, where each agent is linked to every other agent. Hence, the set of neighbors of an agent $i$ is the whole set of agents (excluding itself). Time is measured in Monte Carlo steps, such that one unit of time corresponds to $N$ agent selections for updating.

Although other forms are possible, in the present study, we adopt the following two general functional forms for the probabilities (we use $q(\tau) \equiv q_\tau$ for brevity in the notation),

$$\text{algebraic decay}: \qquad q_\tau = \frac{q_\infty \tau + q_0 \tau^*}{\tau + \tau^*}, \tag{1}$$

$$\text{exponential decay}: \qquad q_\tau = q_\infty + (q_0 - q_\infty) e^{-\tau/\tau^*}, \tag{2}$$

where $q_0$ and $q_\infty$, satisfying $0 \le q_\infty < q_0 \le 1$, denote, respectively, the initial ($\tau = 0$) and asymptotic ($\tau \to \infty$) values of $q_\tau$, and $\tau^*$ is a parameter dictating the rate of decrease of the aging-probability, i.e., a larger value of $\tau^*$ implies a slower decay. Both forms have been considered before in the context of the study of the way that the voter model (without idiosyncratic changes) approaches the asymptotic consensus state [38]. A previous study of the noisy voter model [29] was limited to the algebraic case with $q_0 = 1/2$, $\tau^* = 2$. The aging-less case is recovered, formally taking the limit $\tau^* \to \infty$, implying $q_\tau = q_0$ for all values of the age $\tau$. Our theoretical treatment is rather general, but we adopt the values $q_0 = 1$ and $q_\infty = 0$ in the numerical simulations.

## 3. Noisy Voter Model with Aging

In this Section, we present the results obtained for the noisy voter model with aging. We build over the treatment of Ref. [29] but present a more general one, valid for arbitrary forms of the aging-update probability.

In this model, each agent $i$ is characterized by its binary state variable, $s_i \in \{-1, +1\}$, and its age or residence time in state $s_i$, $\tau_i$. We denote with $x_\tau^\pm(t)$ the fractions of agents in states $\pm1$ and with age $\tau$, such that $x(t) = \sum_{\tau=0}^\infty x_\tau^+(t)$ is the total fraction of agents in state $+1$ at time $t$, and $1 - x(t) = \sum_{\tau=0}^\infty x_\tau^-(t)$ is the total fraction of agents in state $-1$ at time $t$. As detailed in Appendix A.1, it is possible to derive a closed evolution equation for $x(t)$. The procedure starts by writing down rate equations for the densities $x_\tau^\pm(t)$. One then performs an adiabatic approximation that assumes that the evolution of $x_\tau^\pm(t)$ is attached to that of $x(t)$ which evolves at a longer time scale. Under this approximation, the equation for the dynamical evolution of the fraction $x(t)$ of agents in state $+1$ at time $t$ is

$$\begin{aligned} \frac{dx}{dt} &= G^{\mathrm{v}}(x), \\ G^{\mathrm{v}}(x) &= \frac{a}{2}(1-2x) + (1-a)x(1-x)[\Phi^{\mathrm{v}}(x) - \Phi^{\mathrm{v}}(1-x)], \end{aligned} \tag{3}$$

where the function $\Phi^{\mathrm{v}}(x)$ is defined as

$$\Phi^{\mathrm{v}}(x) = \frac{\sum_{\tau=0}^\infty q_\tau F_\tau^{\mathrm{v}}(x)}{\sum_{\tau=0}^\infty F_\tau^{\mathrm{v}}(x)}, \tag{4}$$

with

$$F_0^{\mathrm{v}}(x) = 1, \quad F_\tau^{\mathrm{v}}(x) = \prod_{k=0}^{\tau-1} \gamma_2(q_k\, x, a), \quad \tau \ge 1, \tag{5}$$

and

$$\gamma_n(z, a) = \frac{a}{n} + (1-a)(1-z). \tag{6}$$

The steady state-solutions, $x_{\mathrm{st}}$, are obtained by setting the time derivative of the density $x(t)$ to equal zero, or $G^{\mathrm{v}}(x_{\mathrm{st}}) = 0$, according to Equation (3). Note that, due to its very nature, the adiabatic approximation does provide the correct values of these steady-state solutions. This is because the exact calculation of the steady-state solutions requires that the rates of change of all densities are set to zero. This leads to the condition $G^{\mathrm{v}}(x_{\mathrm{st}}) = 0$, which is also a result of the adiabatic approximation. What the approximate dynamical Equation (3) provides is the stability of the different steady-state solutions as determined by the sign of the derivative $dG^{\mathrm{v}}(x)/dx$ evaluated at $x_{\mathrm{st}}$.

In the aging-less case, $q_\tau = q_0$, $\forall \tau$, it is $\Phi^{\mathrm{v}}(x) = q_0$ and the only fixed point of Equation (3) is $x_{\mathrm{st}} = 1/2$, which is stable. This indicates that the disordered phase is the only asymptotic solution. It is quite clear from the structure of Equation (3) that $x_{\mathrm{st}} = 1/2$ is, trivially, also a steady-state solution for the aging situation, for an arbitrary function $\Phi^{\mathrm{v}}(x)$. The stability of this trivial solution and the existence of other steady-state solutions depend solely on the function $G^{\mathrm{v}}(x)$.

For the algebraic functional form of the aging-update probabilities given by Equation (1), it is possible, as explained in Appendix B, to obtain an analytical form for the function $\Phi^{\mathrm{v}}(x)$ in terms of hypergeometric functions. However, even in this case, the non-trivial steady state values and their stability have to be obtained numerically. For the exponential form of the aging-update probabilities given by Equation (2), it does not seem to be possible to express $\Phi^{\mathrm{v}}(x)$ in terms of other known functions, but we outline in Appendix B.2 a convenient numerical algorithm for its evaluation.

Both for the algebraic and the exponential forms of the aging-update probabilities, the numerical analysis concludes that, in the case that $q_0 = 1$, $q_\infty = 0$, the solution $x_{\mathrm{st}} = 1/2$ is stable for $a > a_c(\tau^*)$ and unstable for $a < a_c(\tau^*)$ and that a new pair of symmetric stable solutions $x_{\mathrm{st}}, 1 - x_{\mathrm{st}}$ emerge for $a \le a_c(\tau^*)$. These two solutions share the same stability, as it can easily be proven that $dG^{\mathrm{v}}(x)/dx|_{x=x_{\mathrm{st}}} = dG^{\mathrm{v}}(x)/dx|_{x=1-x_{\mathrm{st}}}$. This change of stability of the symmetric solution $x_{\mathrm{st}} = 1/2$ is understood as a phase transition between a disordered phase at $a > a_c(\tau^*)$ where both opinions $\pm 1$ coexist in equal proportion to an ordered, consensus phase for $a < a_c(\tau^*)$ where one of the opinions is majoritarian. The existence of this phase transition is one of the main results presented in Ref. [29] for a particular form of the aging probability (corresponding to the algebraic case with $q_0 = 1/2$, $\tau^* = 2$). The same transition was understood as a mapping of the problem with aging to another one with suitable nonlinear rates in the social mechanism [36]. We now present a general study of both the algebraic and exponential dependence of the aging probability and different values on the parameter $\tau^*$.

In Figure 1, we plot the magnetization, $m_{\mathrm{st}} = |2x_{\mathrm{st}} - 1|$, as a function of the noise intensity, $a$, both for the algebraic and the exponential functional forms of the aging update probability and for several values of the parameter $\tau^*$. In the same plot we show the results of numerical simulations of the stochastic rules for the agent-based model detailed in Section 2. The simulations agree remarkably well with the analytical results, validating the adiabatic approximation introduced in the analysis. There are some deviations with respect to the analytical calculation for values of the probability parameter $a$ which are close to the critical value $a_c$. This is attributed to the necessary consideration of a finite number of agents $N = 10^4$ in the numerical simulations, while the theoretical treatment considers the thermodynamic limit $N \to \infty$.

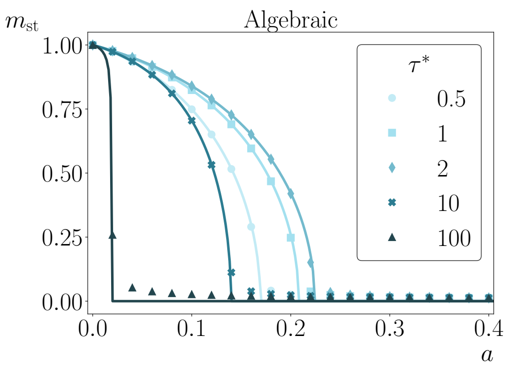


**Figure 1.** Noisy voter model with aging. Magnetization, $m_{\mathrm{st}} = |2x_{\mathrm{st}} - 1|$, as a function of the noise intensity, $a$, for different values of the $\tau^*$ parameter, for the algebraic (**left**) and exponential (**right**) kernels. Solid lines correspond to the theoretical results, while symbols represent the outcomes of numerical simulations with $N = 10^4$ agents, averaged over $5 \times 10^6$ Monte Carlo steps (MCS) after a transient of $5 \times 10^6$ MCS. Notice the non-monotonic dependency of the critical point with the parameter $\tau^*$. See text for details.

For a given value of $\tau^*$, the critical value $a_c$ can be found by calculating the point at which the symmetric solution $x_{\mathrm{st}} = 1/2$ changes from being stable to being unstable, i.e., by solving the equation $dG^{\mathrm{v}}(x)/dx|_{x=1/2} = 0$. A relatively simple calculation shows that this condition is equivalent to

$$(1 - a_{\mathrm{c}}) \left. \frac{d\Phi^{\mathrm{v}}(x)}{dx} \right|_{x=1/2} = 2a_{\mathrm{c}}. \tag{7}$$

Note that $a_{\mathrm{c}}$ also appears in the function $\Phi^{\mathrm{v}}(x)$. The dependence of the critical value $a_{\mathrm{c}}$ on the parameter $\tau^*$, obtained by numerically solving the previous equation, is shown in Figure 2. The larger the value of $a_{\mathrm{c}}$, the larger the region in parameter space in which a consensus-like phase is present. Remarkably, both for the algebraic and the exponential cases, the critical value tends toward zero for $\tau^* \to 0$ and $\tau^* \to \infty$, indicating that in those limits, the only asymptotic solution is the disordered one. The limit $\tau^* \to \infty$ leads to $q_\tau = 1$, $\forall \tau$, which corresponds to the standard version (without aging) of the noisy-voter model. It is known that this model displays a finite-size noise-induced phase transition between disorder and consensus at a critical value of the parameter $a_{\mathrm{c}} = 2/N$. In the thermodynamic limit, it is $a_{\mathrm{c}} \to 0$, in accordance with our result. The reason for the disappearance of the consensus in the limit $\tau^* \to 0$ stems from the fact that the social mechanism is activated with a probability $q(\tau)$ that tends toward 0 for all $\tau$, except $\tau = 0$, for which $q(0) = q_0 > 0$. Hence, the only effective updating mechanism acting at all times is the set of random updates that necessarily lead to disorder. In between these two limits, there is an optimal value $\tau^*_{\mathrm{c}}$ for which $a_{\mathrm{c}}$ is the maximum. For the algebraic case, it is $\tau^*_{\mathrm{c}} = 2.01$, and the corresponding value of the noise intensity is $a_{\mathrm{c}} = 0.224$, while for the exponential case, we find that $\tau^*_{\mathrm{c}} = 3.77$ and $a_{\mathrm{c}} = 0.242$. The critical lines $a_{\mathrm{c}}(\tau^*)$ of the algebraic and exponential cases cross at the point $\tau^*_0 = 2.33$, such that for $\tau^* < \tau^*_0$, the algebraic aging shows a larger consensus region (larger value of $a_{\mathrm{c}}$) and the opposite for $\tau^* > \tau^*_0$.

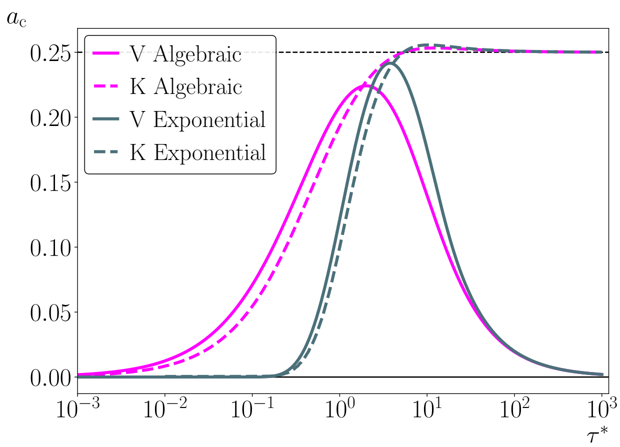


**Figure 2.** Theoretical curves of the critical noise value $a_{\mathrm{c}}$ versus the $\tau^*$ parameter in the algebraic and exponential cases for the noisy voter (V) and kinetic exchange (K) models. The horizontal lines correspond to the critical value in the aging-less case in the noisy voter (solid line) and in the kinetic exchange (dashed line) models.

## 4. Comparison with the Noisy Kinetic Exchange Model with Aging

We now compare the main similarities and differences between the noisy voter model and the kinetic exchange model, both under the presence of aging.

The kinetic exchange model introduces a third, neutral value of the state variable, $s_i \in \{-1, 0, +1\}$. At variance with the noisy voter model, a macroscopic characterization uses both densities $x^{\pm}(t)$ of agents in states $\pm 1$ at time $t$. Via normalization, the density of the agents in state 0 is given by $x^0(t) = 1 - x^+(t) - x^-(t)$.

Along the lines used for the voter model and developed in more detail in Ref. [37], it is possible to write down a closed system of equations for the densities $x^{\pm}(t)$, namely,

$$\begin{aligned} \frac{dx^+}{dt} &= G^{\kappa}(x^+, x^-), \\ \frac{dx^-}{dt} &= G^{\kappa}(x^-, x^+), \\ G^{\kappa}(z,w) &= (1-a)[z(1-z-w)\Phi^{\kappa}(z+w) - z\,w\,\Phi^{\kappa}(w)] + \frac{a}{3}(1-3z). \end{aligned} \tag{8}$$

where

$$\Phi^{\kappa}(x) \equiv \frac{\sum_{\tau=0}^{\infty} q_\tau F_\tau^{\kappa}(x)}{\sum_{\tau=0}^{\infty} F_\tau^{\kappa}(x)}. \tag{9}$$

$$F_0^{\kappa}(x) = 1, \quad F_\tau^{\kappa}(x) = \prod_{k=0}^{\tau-1} \gamma_3(q_k\,x, a), \quad \tau \geq 1, \tag{10}$$

where $\gamma_n(z,a)$ is given by Equation (6). Although the dynamical system for the kinetic model, Equation (8), is quite different from that of the voter model, Equation (3), a quite similar structure is found for the functions $\Phi^{\mathrm{v}}(x)$ (4) and $\Phi^{\kappa}(x)$ (9). As explained in Appendix B, both functions are expressible in terms of hypergeometric functions for the algebraic dependence of the aging probabilities and can be computed using a very efficient numerical algorithm in the case of the exponential dependence.

Let us summarize now the main results of the analysis of the dynamical system Equation (8). In the aging-less case, $\Phi^{\kappa}(x) = q_0$, the dynamical equations always admit the steady-state solution $x_{\mathrm{st}}^+ = x_{\mathrm{st}}^- = 1/3$. This solution becomes unstable for values of the noise intensity $a$ less than the critical value $a_{\mathrm{c}} = q_0/(3+q_0)$, where a pair of non-symmetric solutions $x_{\mathrm{st}}^+ \neq x_{\mathrm{st}}^-$ emerge. The behavior of the magnetization $m_{\mathrm{st}} = |x_{\mathrm{st}}^+ - x_{\mathrm{st}}^-|$ is given by

$$m_{\mathrm{st}} = \begin{cases} \dfrac{\sqrt{(q_0-(3+q_0)a)(3q_0-(1+3q_0)a)}}{\sqrt{3}\,q_0\,(1-a)}, & a \leq a_{\mathrm{c}}, \\ 0, & a \geq a_{\mathrm{c}}, \end{cases} \tag{11}$$

a result already obtained by Nuno Crokidakis [18] in the case that $q_0 = 1$.

When aging is included, the basic structure of the solution remains [37]: there is always a symmetric phase $x_{\mathrm{st}}^+ = x_{\mathrm{st}}^-$ steady-state solution which is stable for $a > a_{\mathrm{c}}(\tau^*)$. For $a < a_{\mathrm{c}}(\tau^*)$, this solution becomes unstable and a pair of non-symmetric stable solutions emerge. It is not possible to give explicit expressions for the solutions $x_{\mathrm{st}}^{\pm}$ for an arbitrary function $\Phi^{\kappa}(x)$, and they have to be determined numerically. The stability of these solutions is also determined numerically from the sign of the eigenvalues of the Jacobian matrix

$$\left.\begin{pmatrix} \dfrac{\partial \dot{x}^+}{\partial x^+} & \dfrac{\partial \dot{x}^-}{\partial x^+} \\[2ex] \dfrac{\partial \dot{x}^+}{\partial x^-} & \dfrac{\partial \dot{x}^-}{\partial x^-} \end{pmatrix}\right|_{x^{\pm}=x_{\mathrm{st}}^{\pm}} \tag{12}$$

evaluated at the fixed points. Here, the dot over a letter denotes the time derivative.

In Figure 3, we show the results of the magnetization $m_{\mathrm{st}}$ as a function of the noise intensity $a$ for two values of the $\tau^*$ parameter for the noisy voter ($m_{\mathrm{st}} = |2x_{\mathrm{st}} - 1|$) and for the kinetic exchange ($m_{\mathrm{st}} = |x_{\mathrm{st}}^+ - x_{\mathrm{st}}^-|$) models with aging. In both cases, we find high agreement between theoretical results and numerical simulations using the stochastic rules of the processes.

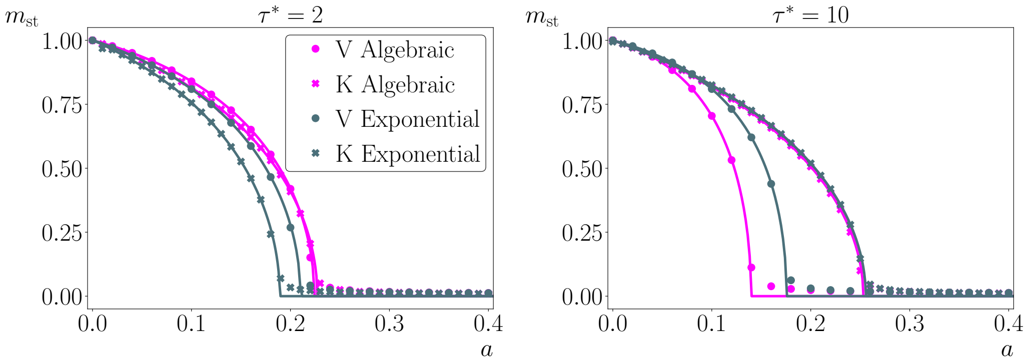


**Figure 3.** Comparison of the noisy voter model (V) and the noisy kinetic exchange model (K), both under the influence of aging. Magnetization $m_{\rm st}$ as a function of the noise intensity $a$ in the algebraic and in the exponential cases for two values of the $\tau^*$ parameter: $\tau^* = 2$ (**left**) and $\tau^* = 10$ (**right**). Results of the mean-field theory are plotted with solid lines, while those of numerical simulations are displayed with symbols. The simulations have been performed with $10^4$ agents and averaged over $5 \times 10^6$ MCS after $5 \times 10^6$ thermalization MCS.

As can be seen in Figure 2, for sufficiently small values of $\tau^*$ (strong or moderate aging, e.g., $\tau^* = 2$ in Figure 3), the most significant determining factor for the existence of a given consensus region is not the type of model considered (either voter or kinetic exchange), but the type of aging kernel. In particular, algebraic aging, which decays most slowly with $\tau$, is the one that most enhances this effect. On the other hand, for large values of $\tau^*$ (weak aging, e.g., $\tau^* = 10$ in Figure 3), the kinetic exchange model gives rise to considerably larger consensus regions than the voter model, regardless of the type of aging considered (algebraic or exponential), which is consistent with the feature that in the aging-less limit ($\tau^* \to \infty$), the kinetic model has a non-null critical point in the thermodynamic limit, in contrast with the voter model.

The most interesting result for the kinetic model is the non-monotonic behavior of the magnetization: the critical value $a_{\rm c}(\tau^*)$ exceeds the value corresponding to the aging-less case $a_{\rm c} = 1/4$, and it then approaches asymptotically to $1/4$. As can be seen in Figure 2, this phenomenon occurs for both profiles of aging. Moreover, as in the noisy voter model with aging, in the kinetic model there is also a value $\tau_0^*$ such that for $\tau^* < \tau_0^*$, the algebraic aging gives rise to a larger consensus region than the exponential aging, while for $\tau^* > \tau_0^*$, the situation is reversed. For the noisy kinetic exchange model, it is $\tau_0^* = 5.40$.

## 5. Final Remarks

We have considered two models of opinion formation, namely the noisy versions of the 2-state voter and 3-state kinetic exchange models, and introduced the effect of aging through the probability of change $q_\tau$ acting on the interaction between agents. Moreover, we considered two different families of kernels for $q_\tau$: either algebraically or exponentially decaying with $\tau$.

A mean-field description has been given for the generic form of $q(\tau)$, unifying the previous results for the two models [29,37] and extending part of them to embrace a wider family of models with $n$ opinion states. This description relies on obtaining the evolution equations for the density of the agents in each opinion state $s$ and with the given age $\tau$. Then, the steady state solutions of this system of (infinite) equations and the analysis of its stability (via an adiabatic approximation valid when separation of timescales holds) allowed for the obtaining of the plots of the order parameter $m_{st}$ vs. the noise probability $a$, in agreement with agent-based simulations in a complete graph.

We have particularly considered the algebraic form $q_\tau = (q_\infty \tau + q_0 \tau^*)/(\tau + \tau^*)$ and the exponential form $q_\tau = q_\infty + (q_0 - q_\infty)\exp(-\tau/\tau^*)$, with $0 \leq q_\infty < q_0 \leq 1$. In numerical examples, we focused on the case that $(q_0, q_\infty) = (1, 0)$. Our results indicate that qualitatively similar tendencies emerge for both kernels in both models as follows.

For sufficiently large $\tau^*$ (weak aging), there is good agreement with the aging-less situation, as expected. For all values of $\tau^*$, the magnetization, $m_{\mathrm{st}}$, as a function of the noise intensity, $a$, displays a continuous transition from order to disorder at a critical value $a_{\mathrm{c}}$. In all the analyzed cases, we observed an optimal value of $\tau^*$ to obtain a consensus (maximizing this critical value $a_{\mathrm{c}}$). It is also noteworthy that, while both models produce different results under weak aging, when aging is strong or moderate, the critical value $a_c$ becomes nearly insensitive to the particular social interaction rule $(s_i, s_j)$ and number of opinion states $n$.

An interesting continuation would be to go beyond all-to-all interactions and to study aging effects on the opinion dynamics in random networks. Moreover, other forms of the aging kernel might also bring new features.

**Author Contributions:** All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** Partial financial support has been received from the Agencia Estatal de Investigación (AEI, MCI, Spain) MICIU/AEI/10.13039/501100011033 and Fondo Europeo de Desarrollo Regional (FEDER, UE) under Project APASOS (PID2021-122256NB-C21) and the María de Maeztu Program for units of Excellence in R&D, grant CEX2021-001164-M. C.A. also acknowledges the partial support received from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq)-Brazil (311435/2020-3) and Fundação de Amparo à Pesquisa do Estado de Rio de Janeiro (FAPERJ)-Brazil (CNE E-26/201.109/2021).

**Data Availability Statement:** Data are contained within the article.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Appendix A. Rate Equations

In the general case of $n$ possible state values $s$, the rate equations for the density $x_\tau^s$ of the agents in state $s$ and age $\tau$ are

$$\frac{dx_\tau^s}{dt} = \Omega_{ss}(\tau-1) - \sum_{s'} \Omega_{ss'}(\tau), \quad \tau \geq 1, \tag{A1}$$

where $\Omega_{ss'}(\tau)$ is the transition rate from state $s$ to state $s'$. The first term accounts for those updates that did not result in a change of state and hence increase their internal time from $\tau - 1$ to $\tau$, while the second accounts for those that resulted in a change of state and therefore resulted in a change $\tau \to 0$ of the internal time. For $\tau = 0$, the rate equation includes all those changes that result in a reset of the internal time,

$$\frac{dx_0^s}{dt} = \sum_{\tau=0}^{\infty} \sum_{s' \neq s} \Omega_{s's}(\tau) - \sum_{s'} \Omega_{ss'}(0). \tag{A2}$$

Adding Equations (A1) and (A2) over all values of $\tau$, one obtains the rate equations for the density of agents in each state $s$ as $x^s = \sum_{\tau=0}^{\infty} x_\tau^s$. We now derive the rate expressions for the noisy voter. A similar treatment for the kinetic exchange model can be found in Ref. [37].

### *Appendix A.1. Noisy Voter Model with Aging*

The state can take two possible values $s \in \{-1, +1\}$. For the all-to-all connectivity adopted in this paper, the set of transition rates $\Omega_{ss'}$ comprises

$$\begin{aligned}
\Omega_{--}(\tau) &= x_\tau^- \left(\frac{a}{2} + (1-a)(1 - x^+ q_\tau)\right), \\
\Omega_{-+}(\tau) &= x_\tau^- \left(\frac{a}{2} + (1-a) x^+ q_\tau\right), \\
\Omega_{+-}(\tau) &= x_\tau^+ \left(\frac{a}{2} + (1-a) x^- q_\tau\right), \\
\Omega_{++}(\tau) &= x_\tau^+ \left(\frac{a}{2} + (1-a)(1 - x^- q_\tau)\right).
\end{aligned} \tag{A3}$$

In this paper, we have used the notation $x$ for $x^+$ and $1-x$ for $x^-$. We show now the derivation of the expression for the first of these rates, $\Omega_{--}(\tau)$: The probability that an agent with internal time $\tau$ and state value $s=-1$ remains $-1$ first requires the selection of an agent in that state (probability $x_\tau^-$); then, if the social rule is chosen (probability $1-a$), the copying mechanism activates with a probability of $q_\tau$ and the selected neighbor must be in the state $-1$ (probability $x^-$), but if the copying mechanism is not activated (probability $1-q_\tau$), the state remains $-1$. If, on the other hand, the idiosyncratic rule is activated (probability $a$) and the new state is chosen to equal $-1$ with a probability of $1/2$, this leads to

$$\Omega_{--}(\tau) = x_\tau^- \left( (1-a)(x^- q_\tau + (1-q_\tau)) + a\frac{1}{2} \right), \tag{A4}$$

which, after the replacement of $x^- = 1-x^+$, leads to the first of Equation (A3). Other transition rates in these equations are obtained similarly.

From these rates and Equations (A1) and (A2), one can derive the rate equations for $x_\tau^s$ as

$$\begin{aligned} \frac{dx_\tau^-}{dt} &= \Omega_{--}(\tau-1) - x_\tau^-, \\ \frac{dx_\tau^+}{dt} &= \Omega_{++}(\tau-1) - x_\tau^+, \end{aligned} \tag{A5}$$

for $\tau \geq 1$, and

$$\begin{aligned} \frac{dx_0^-}{dt} &= \frac{a}{2}x^+ + (1-a)x^- y^+ - x_0^-, \\ \frac{dx_0^+}{dt} &= \frac{a}{2}x^- + (1-a)x^+ y^- - x_0^+, \end{aligned} \tag{A6}$$

for $\tau = 0$, where we have defined

$$y^s \equiv \sum_{\tau=0}^{\infty} q_\tau x_\tau^s. \tag{A7}$$

To obtain a closed equation for the time evolution of the densities $x^s$, we use an adiabatic approximation whereby we assume that the time derivative Equation (A5) of the age-dependent density $x_\tau^s$ can be set to zero, allowing for the expression of $x_\tau^\pm$ in terms of $x_{\tau-1}^\pm$. The solution of this recursive relation is

$$\begin{aligned} x_\tau^- &= x_0^- F_\tau(x^+), \\ x_\tau^+ &= x_0^+ F_\tau(x^-), \end{aligned} \tag{A8}$$

with

$$F_0(x) = 1, \qquad F_\tau(x) \equiv \prod_{k=0}^{\tau-1} \gamma_2(q_k\, x, a), \quad \tau \geq 1, \tag{A9}$$

and the function $\gamma_2$ is defined in Equation (6). Adding Equation (A8) over all $\tau \geq 0$, we obtain

$$\begin{aligned} x^- &= x_0^- \sum_{\tau=0}^{\infty} F_\tau(x^+), \\ x^+ &= x_0^+ \sum_{\tau=0}^{\infty} F_\tau(x^-), \end{aligned} \tag{A10}$$

which, substituted in Equation (A8), leads to

$$\begin{aligned} x_\tau^- &= x^- \frac{F_\tau(x^+)}{\sum_\tau F_\tau(x^+)}, \\ x_\tau^+ &= x^+ \frac{F_\tau(x^-)}{\sum_\tau F_\tau(x^-)}, \end{aligned} \tag{A11}$$

which are now expressed in terms of the global variables $x^\pm$.

Adding Equations (A5) and (A6) over all values of $\tau$, we obtain the corresponding equations for the density $x^s$ of each state $s$.

$$\begin{aligned} \frac{dx^-}{dt} &= \frac{a}{2}(x^+ - x^-) + (1-a)(x^- y^+ - x^+ y^-), \\ \frac{dx^+}{dt} &= \frac{a}{2}(x^- - x^+) + (1-a)(x^+ y^- - x^- y^+). \end{aligned} \tag{A12}$$

Moreover, note that one of the two equations can be eliminated, since $x^+ + x^- = 1$. Then, in order to obtain a closed evolution equation for the global variable $x^+$, one needs to express the variables $y^s$ appearing in Equation (A12) in terms of $x^+$. This can be done by using Equation (A11) into Equation (A7), yielding

$$y^- = (1-x^+)\Phi(x^+), \qquad y^+ = x^+\Phi(1-x^+), \tag{A13}$$

where we have introduced the function

$$\Phi(x) \equiv \frac{\sum_{\tau=0}^\infty q_\tau F_\tau(x)}{\sum_{\tau=0}^\infty F_\tau(x)}. \tag{A14}$$

Let us note here, for consistency, that in the aging-less case, $q_\tau = 1$, it is $\Phi(x) = 1$ and, hence, $y^s = x^s$, for $s = -1, +1$.

The replacement of Equation (A13) in Equation (A12) leads to the rate equation for $x \equiv x^+$, Equation (3). Note that the function $\Phi(x)$ depends on the aging profile $q_\tau$ both through the explicit dependence on Equation (A14) and in the expression of the $F_\tau(x)$ of Equation (A9). As shown in Appendix B, the function $\Phi(x)$ can be related to hypergeometric functions in the case of an algebraic dependence on the aging probability.

## Appendix B. Calculation of Sums Involving $F_\tau(x)$

### *Appendix B.1. Algebraic Aging*

In the case of a rational function of the age of the general form

$$q_\tau = \frac{q_\infty \tau + q_0 \tau^*}{\tau + \tau^*}, \tag{A15}$$

where $\tau^* > 0$ and $0 \le q_\infty < q_0$, the function $F_\tau(x)$ defined is given by

$$F_\tau(x) \equiv \prod_{k=0}^{\tau-1} \gamma_n(q_k\, x, a) = \gamma_n(q_\infty x, a)^\tau \frac{(\tau^* \xi_n(x,a))_\tau}{(\tau^*)_\tau}, \quad \tau \ge 1, \tag{A16}$$

where

$$\xi_n(x,a) \equiv \frac{\gamma_n(q_0 x, a)}{\gamma_n(q_\infty x, a)}, \tag{A17}$$

with the function $\gamma_n(z,a)$ defined in Equation (6), and $(z)_\tau \equiv \Gamma(z+\tau)/\Gamma(z)$ is the Pochhammer symbol and $\Gamma$ is the gamma function.

In order to compute the function $\Phi(x) = \frac{\sum_{\tau=0}^\infty q_\tau F_\tau(x)}{\sum_{\tau=0}^\infty F_\tau(x)}$, one needs the following sums:

$$\sum_{\tau=0}^{\infty} F_\tau(x) = {}_2F_1(1, \tau^* \xi_n(x,a); \tau^*; \gamma_n(q_\infty x, a)), \tag{A18}$$

$$\begin{aligned}\sum_{\tau=0}^{\infty} q_\tau F_\tau(x) =& q_0 \, {}_2F_1(1, \tau^* \xi_n(x,a); 1+\tau^*; \gamma_n(q_\infty x, a)) + \\ & \frac{q_\infty}{1+\tau^*} \gamma_n(q_0 x, a) \, {}_2F_1(2, 1+\tau^* \xi_n(x,a); 2+\tau^*; \gamma_n(q_\infty x, a)).\end{aligned} \tag{A19}$$

In this paper, we cover the algebraic case corresponding to $q_0 = 1$ and $q_\infty = 0$.

*Appendix B.2. Exponential Aging*

In the definition of $F_\tau(x)$, Equation (A16), where $\gamma_n(z,a)$ is given by Equation (6), the substitution of $q_\tau = e^{-\tau/\tau^*}$ yields

$$F_\tau(x) = \Bigg(\underbrace{1 - \frac{n-1}{n} a}_{\alpha}\Bigg)^{\tau} \Bigg(\underbrace{nx \frac{1-a}{n-(n-1)a}}_{b}; e^{-1/\tau^*}\Bigg)_{\tau}, \tag{A20}$$

where $(r;s)_0 = 1$ and $(r;s)_k = \prod_{i=0}^{k-1}(1 - rs^i)$ for $k \geq 1$ is the $q$-Pochhammer symbol, and where $\alpha \in (1/n, 1)$ and $b \in (0,1)$. Therefore, the function $\Phi(x)$ is given by

$$\Phi(x) = \frac{\sum_{\tau=0}^{\infty} q_\tau F_\tau(x)}{\sum_{\tau=0}^{\infty} F_\tau(x)}. \tag{A21}$$

A way to compute numerically the infinite series in the numerator and denominator of Equation (A21) is to cut them off by finite sums up to an upper index $\tau = M$. However, the convergence seems to be slow, and large values of $M$ are needed for a precise calculation, especially for small values of $\tau^*$ or $a$. To overcome this difficulty, an efficient procedure has been described in Ref. [37]. It starts by introducing the function

$$\phi(z, b, s) = \sum_{\tau=0}^{\infty} (b; s)_\tau \, z^\tau, \tag{A22}$$

from which Equation (A21) reads

$$\Phi(x) = \frac{\phi(\alpha s, b, s)}{\phi(\alpha, b, s)}, \quad s = e^{-1/\tau^*} \in (0,1). \tag{A23}$$

Then, we use the iteration relation that follows from its definition, Equation (A22), namely,

$$\phi(\alpha, bs^k, s) = 1 + \alpha(1 - bs^k) \, \phi(\alpha, bs^{k+1}, s), \quad k = L, L-1, \ldots, 0 \tag{A24}$$

with the initial condition

$$\phi(\alpha, bs^{L+1}, s) = \frac{1}{1-\alpha}, \tag{A25}$$

that results from the approximation $bs^{L+1} \approx 0$ and the use of $(0;s)_k = 1$ in the definition (A22). We have taken $L$ such that $bs^{L+1} < \epsilon = 10^{-12}$, or $L \sim \log(\epsilon/b)/\log s = \tau^* \log(b/\epsilon)$, although other smaller values of $\epsilon$ produced the same results.

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